Matrix Analysis: Spectral theorems

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1. Spectral theorems

Outline

Definition 2.1

A matrix $A \in \mathbb{K}^{n \times n}$ *is called* **normal** *is it commutes with its conjugate transpose:* $A^*A = AA^*$.

Example 2.1

This concept generalizes known matrices:

- Symmetric matrices
- Unitary and orthogonal matrices
- Skew-symmetric matrices

Exercise 2.1

Is the following matrix normal? Unitary? Orthogonal? Symmetric? Skew-symmetric?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Conclude that set of normal matrices is a strictly bigger set than the previous examples.

Proposition 2.1

Show that a square matrix A is triangular and normal if and only if A is diagonal.

Theorem 2.1 (Spectral theorem for normal matrices)

Let A be a squared normal matrix. Then there exists an orthonormal basis of eigenvectors.

Remark 2.1

The existence of an orthonormalbasis of eigenvectors is equivalent to the diagonalizability of A by a unitary matrix. Indeed, let $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ be the matrix containing the n orthogonal (and normalized!) eigenvectors with respective eigenvalues $\lambda_1, \dots, \lambda_n$. Then

1. V is unitary (see theorem before Schur's triangularization)
2. $AV = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)V$.

Note that the converse in the previous theorem is obviously true.

Definition 2.2

A is said to be unitarily diagonalizable if it is diagonalizable via a unitary matrix.

Exercise 2.2

Are all diagonalizable matrices unitary diagonalizable?

Corollary 2.1

Let A be a squared normal matrix and let $\mathbf{u}$ and $\mathbf{v}$ be two eigenvectors with distinct eigenvalues $\lambda \neq \mu$. Then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ (where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product).

Remark 2.2

Finding the spectral decomposition of a normal matrix is *easy*:

1. Find the eigenvalues of the matrix
2. Find an orthogonal basis of the eigenspaces $\mathrm{Ker}(A - \lambda)$ for all eigenvalues λ (this can be done by finding linearly independent eigenvectors, and then applying Gram-Schmidt)
3. Stack all the vectors together, keeping the diagonal entries of the diagonal matrix in the right order!

Exercise 2.3

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Diagonalize A and verify the orthogonality of the eigenvectors.

Remark: This is a matrix describing the motion of a system of springs of similar length and resistance, and attached to the same masses.

Exercise 2.4

Find an orthogonal matrix U such that $U^T AU$ is diagonal, where

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Theorem 2.2

Special normal matrices are easy to recognize: Let A be a normal matrix. The followings hold

- A *is Hermitian* ($A^* = A$) *if and only if* $\sigma(A) \subset \mathbb{R}$.
- A *is skew-symmetric* ($A^* = -A$) *if and only if* $\sigma(A) \subset \imath\mathbb{R}$ *(i.e. for all* $\lambda \in \sigma(A), \Re(\lambda) = 0$*)*
- A *is unitary* ($AA^* = I$) *if and only if* $|\lambda| = 1$ *for all* $\lambda \in \sigma(A)$.

Exercise 2.5

Characterize all the normal matrices which are also nilpotent.

Theorem 2.3

Let $A, B \in \mathbb{K}^{n\times n}$ *be two commuting square matrices and assume that* A *is normal. Then* A^* *commutes with* B*.*

Corollary 2.2

Assume that A and B are two normal commuting matrices. Then AB is also normal.

Exercise 2.6

Let $A \in \mathbb{R}^{n\times n}$. Define two quadratic forms $Q_1, Q_2 : \mathbb{R}^n \to \mathbb{R}$ as

$$Q_1(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$
$$Q_2(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(A + A^T)\mathbf{x}.$$

Show that $Q_1(\mathbf{x}) = Q_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

Definition 2.3

Let $A \in \mathbb{K}^{n\times n}$ *be a Hermitian matrix. The scalar function*

$$Q: \begin{array}{ccc} \mathbb{K}^n & \to & \mathbb{K} \\ \mathbf{x} & \mapsto & \langle A\mathbf{x}, \mathbf{x}\rangle \end{array}$$

is called a **complex quadratic form**.

Theorem 2.4

Let $A \in \mathbb{K}^{n\times n}$*. The following statements are equivalent:*

1. A *is Hermitian.*
2. $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$*, for all* $\mathbf{x} \in \mathbb{K}^n$*.*
3. $A^*A = AA^*$ *and* $\sigma(A) \subset \mathbb{R}$*.*
4. S^*AS *is Hermitian for all* $S \in \mathbb{K}^{n\times n}$*.*

Definition 2.4

Let A be a diagonal matrix. The complex quadratic form defined as $Q_A(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle$ is called a **diagonal complex quadratic form***.*

Exercise 2.7

What are the general formula (based on the coordinates of the vectors and entries of the matrices) for a complex quadratic form and a diagonal form?

Proposition 2.2

Every complex quadratic form can be transformed into a diagonal quadratic form.

Proposition 2.3

Let $A \in \mathbb{K}^{n\times n}$ *and let* $\lambda_1, \cdots, \lambda_n$ *be its eigenvalues enumerated such that* $\lambda_1 \leq \cdots \leq \lambda_n$*. (Note: This ordering is possible since the eigenvalues are real!) Then (see homework)*

$$\lambda_1 \|\mathbf{x}\|_2^2 \leq Q_A(\mathbf{x}) \leq \lambda_n \|\mathbf{x}\|_2^2.$$

Exercise 2.8

Let A be a Hermitian matrix. Prove the inequality

$$\lambda_1 \|\mathbf{x}\|_2^2 \leq Q_A(\mathbf{x}) \leq \lambda_n \|\mathbf{x}\|_2^2.$$

Exercise 2.9

When do you get equalities in the previous proposition?

Theorem 2.5 (Courant-Fisher)

Let $A \in \mathbb{K}^{n \times n}$ *be a Hermitian matrix and* $1 \leq k \leq n$*. Then*

$$\lambda_k = \min_{\dim(V)=k} \max_{\mathbf{x} \in V; \|\mathbf{x}\|_2=1} \langle A\mathbf{x}, \mathbf{x} \rangle = \max_{\dim(V)=n-k+1} \min_{\mathbf{x} \in V; \|\mathbf{x}\|_2=1} \langle A\mathbf{x}, \mathbf{x} \rangle.$$

Theorem 2.6 (Weyl)

Let A, B *be two* $n \times n$ *Hermitian matrices. Then*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B).$$