Matrix Analysis: Schur's triangularization

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1. The Jordan canonical form

Outline

1 The Jordan canonical form

Definition 2.1

Two matrices A and B are called **unitary equivalent** *if there exists a unitary matrix U such that*

$$A = UBU^*.$$

We will write this as $A \approx B$.

Exercise 2.1

Let A and B be two matrices such that $A \approx B$. Show that

$$\|A\|_F = \|B\|_F.$$

Theorem 2.1 (Schur's triangularization)

Let $A \in \mathbb{K}^{n\times n}$ *with (repeated, potentially commplex) eigenvalues* $\lambda_1, \cdots, \lambda_n$. *Then* A *is unitarily equivalent to an upper triangular matrix* T *whose diagonal entries are* $t_{i,i} = \lambda_i$*: there exists a unitary matrix* U *such that*

$$A = U \begin{bmatrix} \lambda_1 & x & \cdots & x \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} U^*.$$

Exercise 2.2

Show the following statements

1. A similar statement is valid in which the triangular matrix is lower.
2. Such a decomposition is not unique.

Exercise 2.3

Let $A \in \mathbb{K}^{n \times n}$ with eigenvalues (counting multiplicities) $\lambda_1, \cdots, \lambda_n$. Show that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{i,j}|^2.$$

Theorem 2.2 (Cayley-Hamilton)

Let $A \in \mathbb{K}^{n\times n}$ *and* $p_A(t)$ *its characteristic polynomial. Then*

$$p_A(A) = 0.$$

Exercise 2.4

Let A be a matrix, S be a non-singular matrix and p a polynomial. Then

$$p(S^{-1}AS) = S^{-1}p(A)S.$$

Conclude that if two matrices are equivalent, then so are all matrices created by applying the same polynomial to A and B.

Exercise 2.5

Carry out the actual computations in the previous proof.

Theorem 2.3

For all $A \in \mathbb{K}^{n \times n}$ *and for all* $\varepsilon > 0$, *there exists a diagonalizable matrix* B *such that*

$$\|A - B\|_F < \varepsilon. \tag{2.1}$$

Theorem 2.4

Given $A \in \mathbb{K}^{n \times n}$ *with distinct eigenvalues* $\lambda_1, \cdots, \lambda_r$*, there exists a non-singular matrix* S *such that*

$$A = S \,\mathrm{diag}(T_1, \cdots, T_r) S^{-1},$$

where T_i *are upper triangular matrices such that*

$$T_i = \begin{bmatrix} \lambda_i & * & \cdots & * \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix},$$

for $1 \leq i \leq r$.

Lemma 1

Let $A \in \mathbb{K}^{m\times m}$ *and* $B \in \mathbb{K}^{n\times n}$ *be two matrices such that* $\sigma(A) \cap \sigma(B) = \varnothing$. *Then for any choice of* $M \in \mathbb{K}^{m\times n}$

$$\begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$