Matrix Analysis: Schur's triangularization

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The Jordan canonical form
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Definition 2.1

Two matrices $A$ and $B$ are called unitary equivalent if there exists a unitary matrix $U$ such that

$$A = UBU^*.$$

We will write this as $A \approx B$. 
Exercise 2.1

Let $A$ and $B$ be two matrices such that $A \approx B$. Show that

$$
\|A\|_F = \|B\|_F.
$$
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Theorem 2.1 (Schur's triangularization)

Let $A \in \mathbb{K}^{n \times n}$ with (repeated, potentially complex) eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $A$ is unitarily equivalent to an upper triangular matrix $T$ whose diagonal entries are $t_{i,i} = \lambda_i$: there exists a unitary matrix $U$ such that

$$A = U \begin{bmatrix}
\lambda_1 & x & \cdots & x \\
0 & \lambda_2 & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & \cdots & 0 & \lambda_n
\end{bmatrix} U^*.$$
Exercise 2.2

Show the following statements

1. A similar statement is valid in which the triangular matrix is lower.
2. Such a decomposition is not unique.
Exercise 2.3

Let $A \in \mathbb{K}^{n \times n}$ with eigenvalues (counting multiplicities) $\lambda_1, \cdots, \lambda_n$. Show that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{i,j}|^2.$$
Theorem 2.2 (Cayley-Hamilton)

Let \( A \in \mathbb{K}^{n \times n} \) and \( p_A(t) \) its characteristic polynomial. Then

\[
p_A(A) = 0.
\]
Exercise 2.4

Let $A$ be a matrix, $S$ be a non-singular matrix and $p$ a polynomial. Then

$$p(S^{-1}AS) = S^{-1}p(A)S.$$  

Conclude that if two matrices are equivalent, then so are all matrices created by applying the same polynomial to $A$ and $B$. 
Exercise 2.5

Carry out the actual computations in the previous proof.
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**Theorem 2.3**

For all $A \in \mathbb{K}^{n \times n}$ and for all $\varepsilon > 0$, there exists a diagonalizable matrix $B$ such that

$$\|A - B\|_F < \varepsilon. \quad (2.1)$$
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**Theorem 2.4**

*Given* $A \in K^{n \times n}$ *with distinct eigenvalues* $\lambda_1, \cdots, \lambda_r$, *there exists a non-singular matrix* $S$ *such that*

$$A = S \text{ diag}(T_1, \cdots, T_r) S^{-1},$$

*where* $T_i$ *are upper triangular matrices such that*

$$T_i = \begin{bmatrix}
\lambda_i & * & \cdots & * \\
0 & \lambda_i & \ddots & \\
& \ddots & \ddots & * \\
0 & \cdots & 0 & \lambda_i
\end{bmatrix},$$

*for* $1 \leq i \leq r$. 

Lemma 1

Let $A \in \mathbb{K}^{m \times m}$ and $B \in \mathbb{K}^{n \times n}$ be two matrices such that $\sigma(A) \cap \sigma(B) = \emptyset$. Then for any choice of $M \in \mathbb{K}^{m \times n}$

$$\begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \sim \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$