Matrix Analysis: Jordan canonical form

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Example 1.1

Study the characteristic polynomials, trace, determinants, eigenvalues of the two following matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute also A^2 and B^2 and conclude that, although these matrices share common characteristics, they cannot be equivalent!

Definition 1.1

A Jordan block of order k and value λ is defined as the matrix

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}, k \geq 2$$

and $J_1(\lambda) = \lambda$.

Exercise 1.1

Compute the following

1. $J_k(0)J_k(0)^T$.
2. $J_k(0)^T J_k(0)$.
3. $I - J_k(0)^T J_k(0)$.

Definition 1.2

We call a matrix a **Jordan matrix**, *a matrix that block-diagonal where each block is itself a Jordan block.*

Theorem 1.1

Let $A \in \mathbb{K}^{n\times n}$. There exists a non-singular matrix S such that

$$A = S\,\mathrm{diag}\left(J_{n_1}(\lambda_1), \cdots, J_{n_k}(\lambda_k)\right) S^{-1},$$

where $n_1 + \cdots + n_k = n$ and $\lambda_1, \cdots, \lambda_k$ are not necessarily distinct eigenvalues of A. The Jordan form is unique, up to permutation of the blocks.

Proposition 1.1

To compute a Jordan canonical form of a matrix A it suffices to follow these steps:

1. *Compute the distinct eigenvalues of A: $\lambda_1, \cdots, \lambda_r$. They have algebraic multiplicities $p_1, \cdots, p_r$.*
2. *Compute $n_i^{(k)} = rk(A - \lambda_i I)^k$ for $1 \leq i \leq r$ and $0 \leq k \leq p_i$ (you can actually stop before p_i: as soon as $n_i^{(k)} = p_i$)*
3. *For each eigenvalue λ_i, they are $n_i^{(k-1)} - n_i^{(k)}$ Jordan-blocks of size $\geq k$. (hence, in the computation above, you can stop once you reach $n_i^{(k)} = n_i^{(k-1)}$)*
4. *Working backwards gives the number of blocks of a given size.*

Example 1.2

Find the Jordan canonical form of the matrix A, given the following information:

$$p_A(\lambda) = (\lambda - 2)^7 (\lambda - 3)^3,$$

$$rk(A - 2I) = 7, \quad rk(A - 2I)^2 = 4, \quad rk(A - 2I)^3 = 3, \quad rk(A - 2I)^4 = 3$$

$$rk(A - 3I) = 8, \quad rk(A - 3I)^2 = 7, \quad rk(A - 3I)^3 = 7.$$

Exercise 1.2

Find the Jordan canonical form of the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 1.1

General strategy for finding the *Jordanizing matrix* (this is far from rigorous ... understanding why this work is no easy thing)

1. Find a set of linearly independent eigenvectors – this gives the number of Jordan blocks
2. Find $\mathbf{v}_i^{(2)}$ such that $(A - \lambda_i I)\mathbf{v}_i^{(2)} = \mathbf{v}_i^{(1)}$ (i.e. find a way to create your eigenvector in the range of $A - \lambda_i$). We say that $\mathbf{v}_i^{(2)}$ is a **(generalized) eigenvector of order** 2 with eigenvalue λ_i.
3. Iterate until you're happy (i.e. you have enough linearly independent vectors, with the appropriate property)

Remark 1.2

They are many different use case: practice as much as you can until you have encountered all of them: e.g.:

- n distinct eigenvalues
- 1 eigenvalue of (algebraic) multiplicity n
- 1 eigenvalue with algebraic multiplicity 3 but geometric multiplicity 2 (how to pick the eigenvectors?)
- algebraic multiplicity 4, geometric multiplicity 2: blocks of size 1 and 3, or 2 and 2 ?
- etc ...

Exercise 1.3

Find the Jordan canonical form and the (generalized) eigenvectors of the following matrices

$$A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 0 & 1 \\ 3 & 7 & 3 \\ -1 & 0 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Regarding C: Make sure to find the generalized eigenvector of order 2 as linear combination of the traditional eigenvectors.

Example 1.3

A: We have that $p_A(t) = -t(1-t)^2$. So we have two eigenvalues: $\lambda = 1$ with algebraic multiplicity 2 and $\mu = 0$ with algebraic multiplicity 1. Let $\mathbf{u}$ be an eigenvector with eigenvalue μ: $A\mathbf{u} = 0$ which means that $\mathbf{u} = \alpha(0, -1, 2)^T$. Let us now look at the eigenvectors with eigenvalue μ. We obtain: $(A - \mu I)\mathbf{v} = 0 \Leftrightarrow \exists \alpha \in \mathbb{K} : \mathbf{v} = \alpha(1, -1, 5)^T$ From this, we obtain two things:

1. Since the dimension of the kernel is one, we only have one Jordan block. Because, λ has multiplicity 2, this block can only have order 2.
2. We have found **one** eigenvector. We still need a generalized eigenvector of order 2, i.e., we want to find a vector $\mathbf{w}$ such that $A\mathbf{w} = \lambda\mathbf{w} + \mathbf{v}$.

This is done by solving the system

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

A set of solutions is given by $\mathbf{w} = (t, 3-t, 5(t-1))^T$, for a parameter $t \in \mathbb{K}$. Particular (smart) examples include $t = 0, 1, 3$ but any value would do. We finally arrive at the following decomposition:

$$A = SJS^{-1},$$

with

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and } S = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$$

Example 1.4

B: This is a diagonal matrix, its characteristic polynomial reads $p_B(\lambda) = (2-\lambda)^5$ hence $\lambda = 2$ is the unique eigenvalue with (algebraic) multiplicity 3. $B - \lambda I = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and therefore, $\mathrm{null}(B - \lambda I) = \dim(\ker(A - \lambda I)) = 3$ and we can expect 3 Jordan blocks. We expect a total order of 5 with 3 blocks. Here are the following options:

1. 1 block of size 1, two of size 2,
2. 2 blocks of size 1 and one of size 3.

A direct computation gives $\mathrm{rank}(B - \lambda I)^2 = 1$. This means that we expect only one block (: 2-1) of size at least 2. This means that the J matrix is composed of two block of size 1 and one of size 3. The eigenvectors are multiples of the basis of the kernel and one of those vectors generates eigenvectors of order 2 and 3.

Basis of the kernel:$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T \in ker(B - \lambda I)$ iff $\mathbf{x} = x_1(1,0,0,0,0)^T + x_2(0,1,-2,0,0)^T + x_5(0,0,0,0,1)^T$ with $x_1, x_2, x_5 \in \mathbb{K}$ (we will call these vectors respectively $\mathbf{u}, \mathbf{v}, \mathbf{w}$).

We still need to find the generalized eigenvectors of order 2 and 3 for one of those eigenvectors. Observing the matrix $B - \lambda I$ shows that we will have

Example 1.5

C: We first compute the characteristic polynomial (we will expand with respect to the second column):

$$p_C(\lambda) = \det\left(\begin{bmatrix} 8-\lambda & 0 & 1 \\ 3 & 7-\lambda & 3 \\ -1 & 0 & 6-\lambda \end{bmatrix}\right) = (7-\lambda)\det\left(\begin{bmatrix} 8-\lambda & 1 \\ -1 & 6-\lambda \end{bmatrix}\right) = (7-$$

Hence the unique eigenvalue is $\lambda = 7$ with algebraic multiplicity 3. Let us find a basis for the kernel:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \ker(A - \lambda I) \Leftrightarrow \mathbf{x} = x(1,0,-1)^T + y(0,1,0)^T.$$

Let these two vectors be called $\mathbf{u}$ and $\mathbf{v}$. They are linearly independent, hence we have 2 Jordan blocks. Since we live in a 3 dimensional world, the only option is to have a block of size 1 and another one of size 2. Unfortunately, we cannot find a vector $\mathbf{w}$ such that $(A - \lambda I)\mathbf{w} = \mathbf{u}$ or $\mathbf{v}$ as we did before. We will therefore have to adapt the basis of eigenvectors that we have found, such that we can find two linearly independent eigenvectors, one of which should in the range of $(A - \lambda I)$. We can keep $\mathbf{v}$ as it is and consider the vector $\mathbf{x} = \alpha\mathbf{u} + \beta\mathbf{v}$ and we want to find α and β, such that

$$(A - \lambda I)\mathbf{x} = 0$$

Exercise 1.4

Let P be a matrix such that $P^2 = P$. Compute its Jordan canonical form.

Definition 1.3

Let A be a $n \times n$ square matrix. The monic polynomial of minimum degree $m_A(x)$ such that $m_A(x) = 0$ is called its **minimal polynomial**.

Remark 1.3

- This polynomial exists: the set of (monic) polynomials which annihilate A is non-void.
- It has degree at most n.
- It is unique!

Theorem 1.2

Let $A \in \mathbb{K}^{n \times n}$ *and* m_A *its minimal polynomials. Let* q *be a polynomial such that* $q(A) = 0$*. Then* m_A *is a divisor of* q*.*

Exercise 1.5

Find the minimal polynomial of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Exercise 1.6

Find the minimal and characteristic polynomials of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 1.7

Let $A \in \mathbb{K}^{n \times n}$ be such that $A^2 = A$. Give its minimal polynomial and compute its Jordan form.

Theorem 1.3

For any n^{th} degree (monic) polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$$

there exists an $n \times n$ matrix A such that $m_A(x) = p(x)$.

Definition 1.4

The A matrix introduced in the previous theorem is called the **companion matrix** *of the monic polynomial.*

Proposition 1.2

Let $X' = AX$ be a linear first order differential system. The general solution is given by

$$X(t) = \sum_{i=1}^{k} X_i(t),$$

where k is the number of Jordan blocks of the matrix A and X_i is the solution restricted to the i^{th} block. It is given by

$$X_i(t) = \sum_{j=1}^{n_i} e^{\lambda t} C_j \sum_{\ell=1}^{j} \frac{t^{j-\ell}}{(j-\ell)!} \mathbf{v}_\ell,$$

where $\mathbf{v}_i$ are the Jordanizing vector of the current block.
For instance, we have:

$$X(t) = C_1 e^{\lambda t} \mathbf{v}_1 + C_2 e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) + C_3 e^{\mu t} \mathbf{v}_3,$$

for a 3×3 problem where one of the eigenvalue has a lack of geometric multiplicity.

Exercise 1.8

Find solutions to the following problems:

$$\begin{cases} x'(t) & = & 2x - 3y \\ y'(t) & = & -x + 4y \end{cases}$$

$$\begin{cases} x'(t) & = & 2x - y \\ y'(t) & = & x + 4y \end{cases}$$

$$y'' = -3y' - 2y.$$