SUGGESTED EXERCISES: MATRIX ANALYSIS

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1. Review of linear algebra

Homework 1. Show that $tr(A^*A) = 0 \Leftrightarrow A = 0$.

Homework 2. True or false (and justify) the following sets are multiplicative groups

- $G_1 = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$.
- $G_2 = \{A \in \mathbb{R}^{n\times n} : |\det(A)| = 1\}$.
- $G_3 = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0 \text{ and } a_{i,j} \in \mathbb{N}\}$.
- $G_4 = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0 \text{ and } a_{i,j} \in \mathbb{Z}\}$.

Homework 3. Given $A \in \mathbb{K}^{m\times n}$, prove or disprove the following statements (remember that A^* is the adjoint of A, i.e. its conjugate transpose – this is equivalent to the transpose of a matrix in case $\mathbb{K} = \mathbb{R}$).

(1) $rk(A^*) = rk(A)$.
(2) $\dim(ker(A)) = \dim(ker(A^*))$.
(3) $ker(A) = ker(A^*A)$.
(4) $\dim(ker(A)) = \dim(ker(A^*A))$.
(5) $rk(A) = rk(A^*A)$.

Homework 4. Prove that $AB = 0 \Leftrightarrow R(B) \subset ker(A)$.

Homework 5. (1) Let A and B be two square matrices. Assume that A is invertible. Show that $p_{AB}(x) = p_{BA}(x)$.
(2) Assume both A and B are singular. Show that $p_{AB}(x) = p_{BA}(x)$.
(3) What happens in case $A \in \mathbb{K}^{m\times n}$ and $B \in \mathbb{K}^{n\times m}$.

Homework 6. Let $A \in \mathbb{K}^{n\times n}$ and let p denotes any polynomial. Show that if $(\lambda, \mathbf{x})$ is an eigenpair of A then $(p(\lambda), \mathbf{x})$ is an eigenpair of $p(A)$. Use this to prove the theorem of Cayley-Hamilton in case of diagonalizable matrices: $p_A(A) = 0$. (You may also prove the theorem in a more general form, i.e. not necessarily when A is diagonalizable. In this case, you can use the fact that matrices over $\mathbb{C}$ which are diagonalizable form a dense subset of $\mathbb{C}^{n\times n}$ and then use this to conclude about matrices with entries in $\mathbb{R}$.)

Homework 7. (1) Let $A, B \in \mathbb{K}^{n\times n}$. Show that $AB = I \Leftrightarrow BA = I$.
(2) Show that this is no longer true in case $A \in \mathbb{K}^{m\times n}$ and $B \in \mathbb{K}^{n\times m}$.

Homework 8. Diagonalize (if possible) the following matrices and factor their characteristic polynomials in $\mathbb{R}$ and $\mathbb{C}$.

(1) $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$.
(2) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.
(3) $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.
(4) $A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$.

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Homework 9 (Beginning of Exercises for Week 2). Assume A and B are two non-singular matrices. Prove that

$$adj(AB) = adj(B).adj(A).$$

Homework 10. Let $A \in \mathbb{R}^{n\times n}$, for $n \geq 2$. Prove that

$$adj(adj(A)) = (\det(A))^{n-2} A.$$

Homework 11. Let S be a subset of a vector space V.

(1) What can be said about $\dim(\text{span}(S))$?
(2) Let $V = C^\infty(\mathbb{R}, \mathbb{R})$ the set of infinitely differentiable functions. Is V finite or infinite dimensional?

Homework 12. Consider the following three basis of $V = \mathbb{R}_2[x]$:

- $S = (x \mapsto 1, x \mapsto x, x \mapsto x^2)$,
- $T = (x \mapsto 1, x \mapsto 1 + x, x \mapsto 1 + x^2)$,
- $U = (x \mapsto x^2 + 1, x \mapsto 1 + x, x \mapsto x + x^2)$,

Consider the following mapping:

$$M : \begin{array}{ccc} \mathbb{R}_2[x] & \to & \mathbb{R}_2[x] \\ p & \mapsto & p' + Xp' \end{array}$$

(1) Show that M is a linear transformation.
(2) Compute its matrix representations when looking at is using all the different basis (i.e. U for both input and output spaces, T for both input and output spaces, and then U).
(3) Show that all these matrices are similar to each other. What is the P matrix appearing in the equivalence
(4) Compute now the matrix of this linear transformation when the input and output basis are not the same. (i.e. 6 matrices in total). Show that for all of these matrices, there exists a pair of non-singular matrices P and Q such that $A = QBP^{-1}$ (where A is one of those matrices and B is another one). What do P and Q correspond to?

Homework 13. Let $A = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Fow which values of a, b, c is A invertible?

Homework 14. Let $(x_i)_{i=1}^n$ be n numbers in $\mathbb{R}$. Let A be the matrix such that $a_{i,j} = x_i^{j-1}$. Show that A is invertible $\Leftrightarrow x_i \neq x_j$ for $i \neq j$.

Homework 15. Let $A \in \mathbb{R}^{n\times n}$. Assume that for all $1 \leq i \leq n$, $\sum_{j=1}^n a_{i,j} = 1$. Prove that $\lambda = 1$ is an eigenvalue and give one of its eigenvectors.

Homework 16. Let $t \in \mathbb{R}$ and let $A = \begin{bmatrix} 1 & t & t & \cdots & t \\ t & 1 & t & \cdots & t \\ t & \vdots & \ddots & \cdots & t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t & t & t & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$. Find the determinant, eigenvalues, eigenvectors of the matrix A and diagonalize it.

Homework 17. Let $A \in \mathbb{K}^{n\times n}$ and assume A is diagonalizable.

(1) Compute, using a power series, $exp(A)$. Verify, by analyzing the continuity of the partial sums, that this operation is well-defined!
(2) Does it hold that $\exp(A+B) = \exp(A)\exp(B)$. If yes, prove it, if no, give an example and a condition for the formula to be true.
(3) Assume moreover $A^3 = 0$. What is $\exp(A)$?
(4) What is $\exp(A^T)$?
(5) Assume $(\lambda, \mathbf{x})$ is an eigenpair of A. What can be said about the eigenvalues and / or eigenvectors of $\exp(A)$?

Homework 18. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{K}^n$.

(1) What is $rk(\mathbf{x}\mathbf{y}^*)$?
(2) Let $A = \begin{bmatrix} 0 & \mathbf{x} \\ \mathbf{y}^* & a \end{bmatrix}$, where $a \in \mathbb{K}$. Compute the characteristic polynomial of A. Show that $rk(A) \leq 2$.

Homework 19. Let $A, B, S \in \mathbb{K}^{n\times n}$ with S non-singular. Show that $AB = BA$ if and only if $S^{-1}AS$ commmutes with $S^{-1}BS$.

Homework 20. Let A and B be two diagonalizable matrices. Show that $AB = BA$ if and only if A and B are simultaneously diagonalizable (i.e. via the same basis).

Homework 21. Let $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$. Show that $p_{A+tI}(\lambda) = p_A(\lambda - t)$. How do the eigenvalues of $A + tI$ relate to those of A?

Homework 22. Let $A \in \mathbb{K}^{n\times n}$.

(1) Let $\lambda \in \sigma(A)$ with multiplicity (geometric AND algebraic) 1. Show that $rk(A - \lambda I) = n - 1$.
(2) Conversely, if $rk(A - \lambda I) = n - 1$, is λ an eigenvalue of A? If yes, does it necessarily have (which?) multiplicity 1?

Homework 23. Let $A \in \mathbb{K}^{n\times n}$. Show that its characteristic polynomial reads

$$p_A(\lambda) = -\lambda^3 + tr(A)\lambda^2 - tr(adj(A))t + \det(A).$$

Homework 24. Prove the inequality between the ℓ_1, ℓ_2 and ℓ_∞ norms.

Homework 25. Let $V = \{f : [0, 1] \to \mathbb{R}\}$. Show that the following defines an inner product:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

Homework 26. Is the following matrix diagonalizable?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Compute its eigenvalues and eigenvectors.

Let a_i represent the i^{th} column of A. If the columns of A are linearly independent, use Gram-Schmidt to transform the matrix A into an orthonormal basis. If they are not, reduce the set of columns to a linear independent set, then extend it to a basis of $\mathbb{R}^3$ and then orthonormalize it.

Homework 27. Let $\mathbf{x} = (1, 2, 3, 0)^T$, $\mathbf{y} = (1, 2, 0, 0)^T$, and $\mathbf{z} = (1, 0, 0, 1)^T$ be three vectors in $\mathbb{R}^4$. Expand this family of vector into a basis and orthonormalize it.

Homework 28. Let $x_0, \cdots x_n$ be $n + 1$ distinct points in $\mathbb{R}$ and consider $V = \mathbb{R}_n[x]$ the set of polynomials of degree up to n.

Show that, given $p, q \in V$,

$$\langle p, q\rangle = \sum_{i=0}^{n} p(x_i)q(x_i)$$

defines an inner product.

Homework 29. Perform Gram Schmidt on the following family of vectors: $\mathbf{u} = [6, 3, 2]^T, \mathbf{v} = [6, 6, 1]^T, \mathbf{w} = [1, 1, 1]^T$.

2. The Jordan canonical form

Homework 30. Let A and B be two given $n \times n$ matrices. Assume that A and B are simultaneously to triangular matrices (i.e. there exists a single invertible matrix S such that $S^{-1}AS$ and $S^{-1}BS$ are upper triangular). Show that all the eigenvalues of $AB - BA$ must be 0.

Homework 31. Let $A \in \mathbb{K}^{n\times n}$. Assume that there exists a $k \geq 1$ such that $A^k = 0$. Show that all the eigenvalues must be 0.

Homework 32. Let $A \in \mathbb{K}^{m\times n}$ be the block matrix defined as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

where $A_{1,1} \in \mathbb{K}^{n\times n}$ and $A_{2,2} \in \mathbb{K}^{m\times m}$. Show that there A is nilpotent if and only if $A_{1,1}$ and $A_{2,2}$ are nilpotent.

Hint: you may want to prove that the eigenvalues of a nilpotent matrix must be 0. It follows from this that (up to a change of basis), $Ae_k \in \operatorname{span}(e_1, \cdots, e_{k-1})$. Conclude from this where $A^j e_k$ may lie (where e_k is the kth basis vector)

Homework 33. Compute the Jordan canonical form and the *Jordanizing* basis of the matrix

$$A = \begin{bmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{bmatrix}.$$

Homework 34. Compute the Jordan canonical form, as well as the generalized eigenvectors (those that lead to the JCF) of the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & -1 & 0 \\ 0-1 & -2 & \\ 0 & 0 & -1 \end{bmatrix}.$$

What are A and B raised to the power 3 and 4?

Homework 35. Let $A \in \mathbb{K}^{n\times n}$ be a matrix such that $|\lambda| < 1$ for all eigenvalues $\lambda \in \sigma(A)$. Show that

$$\lim_{k\to\infty} A^k = 0.$$

(this is important in proving convergence of certain stochastic processes, which are used, among others, in mathematical genetics.)

Homework 36. Why you should not use JCF on a computer.

Let B_ε be the matrix defined with a parameter $\varepsilon > 0$ as

$$B_\varepsilon = \begin{bmatrix} 1 & \varepsilon & 0 \\ 0 & 1 & 0 \\ \varepsilon & 0 & 1 \end{bmatrix}.$$

Let J_ε be its canonical Jordan form and let $J = J_0$. Compare $\|B_0 - B_\varepsilon\|_F$ and $\|J - J_\varepsilon\|_F$ and conclude that *Jordanizing* a matrix is an unstable process.

Homework 37. Find the minimal polynomial of the following matrix

$$A = \begin{bmatrix} -14 & 3 & -36 \\ -20 & 5 & -48 \\ 5 & -1 & 13 \end{bmatrix}$$

Homework 38. Let $A \in \mathbb{K}^n$ such that $p_A(x) = m_A(x)$ (the characteristic polynomial is also the minimal one). What can be said about the Jordan canonical form of this matrix?

Homework 39. Solve the following initial value problem:

$$X'(t) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Homework 40. Find real solutions to the following second-order differential equation, using the matrix formulation.

$$y'' + y = 0, \quad 0 \le t \le 2\pi$$

Homework 41. Find the general solutions to the following system of differential equations

$$\begin{cases} x' &= -7x-5y-3z, \\ y' &= 2x-2y-3z, \\ z' &= y. \end{cases}$$

3. Spectral theorems

Homework 42. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Diagonalize (with unitary/orthogonal similarity) A.

Homework 43. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Diagonalize (with unitary/orthogonal similarity) A.

Homework 44. Let A, B, C, D be four matrices such that AD and BC are Hermitian.

(1) Show that A and D are not necessarily Hermitian (and so do C and B!).
(2) Show that $AD - C^*B^* = I \Rightarrow DA - BC = I$.

Homework 45. Let $A = C + iB \in \mathbb{C}^{n\times n}$, with $C, B \in \mathbb{R}^{n\times n}$. Show the following

$$A \text{ is Hermitian } \Leftrightarrow C \text{ is symmetric and } B \text{ is skew-symmetric.}$$

Homework 46. Prove the second equality in Courant-Fisher's theorem.

Homework 47. Given two Hermitian matrices A and B. Show that A and B are similar if and only if they are unitarily similar.

Homework 48. Let A and B be two Hermitian matrices and assume that $A - B$ has only nonnegative eigenvalues. Show that

$$\lambda_i(A) \geq \lambda_i(B).$$

Homework 49. This exercise deals with the approximation of Hermitian matrices. This will be important in the next chapter of the course.

Let A be an $n \times n$ Hermitian matrix.

(1) Justify the existence of a decomposition $A = U\Lambda U^*$, where Λ is a real diagonal matrix and $U = (\mathbf{u}_1, \cdots, \mathbf{u}_n)$. We will assume that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.
(2) Show that $A = \sum_{i=1}^n \lambda_i \mathbf{u}_i \mathbf{u}_i^T$.
(3) Let $A_k = \sum_{i=1}^k \lambda_i \mathbf{u}_i \mathbf{u}_i^T$. Show that for any $\mathbf{x} \in \mathbb{K}^n$ with $xbf\|_2 = 1$ we have $\|(A - A_k)\mathbf{x}\|_2^2 \leq \sum_{i=k+1}^n |\lambda_i|^2$.
(4) Prove that $\|A - A_k\|_{2\to 2} := \max_{\|\mathbf{x}\|_2=1} \|(A - A_K)\mathbf{x}\|_2 \leq \left(\sum_{i=k+1}^n |\lambda_i|^2\right)^{1/2}$.
(5) Find the rank 2 and 3 approximations based on the spectral decomposition of the following matrix and estimate the errors in the $\|\cdot\|_{2\to 2}$ norm.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

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