EXERCISE SHEET 3: MATRIX ANALYSIS

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Remember: This is a mandatory exercise.
Turn in your homework no later than **Sunday, December 9th** at the beginning of class.
You may work in group, but the final version should be your own writing.
Papers that are too obviously written from someone else's will not be graded.

The following rules apply:

- Only 10 of the exercises will contribute to your grade.
- The first 6 exercises are mandatory.
- The remaining 4 graded exercises can be picked from the list.
- Indicate clearly at the beginning of your paper which exercises should contribute to your mark. Should this information not appear clearly, the first 10 will be marked.
- The homework are too be turned in individually (preparation can be done in groups).

Homework 1. Let S_a be the set of arithmetic progressions (i.e. sequences $(u_n)_{n\in\mathbb{N}}$ for which there exists an $r \in \mathbb{R}$ such that $u_{n+1} - u_n = r$). Show that

(1) S_a is a vector space.
(2) The mapping $T : S_a \to \mathbb{R}, (u_n)_n \mapsto r$ is a linear transformation.

Homework 2. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map. Show that $ker(f) \neq \{0\}$.

Homework 3. Let A be a real non-singular $n \times n$ matrix. Show that

$$p(\mathbf{u}, \mathbf{v}) := \mathbf{u}^T A^T A \mathbf{v}$$

defines an inner product.

Homework 4. (1) Prove that the following family of vectors is linearly independent. $\mathbf{x} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y} := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{z} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Use Gram-Schmidt orthonormalization process on this family to generate an orthonormal basis of $\mathbb{R}^3$ if the inner products are given by (prove first that they are inner products ... you may re-use, without further justifications, some exercises from this sheet or from class)

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_i u_i v_i$.
(b) $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = \sum_i i^2 u_i v_i$.

Homework 5. Let A and B be two $n \times n$ matrices over $\mathbb{R}$ or $\mathbb{C}$ such that $AB = BA$. Assume moreover that there exists a $p \leq n$ such that $C^p = 0$ with $C = A - B$.

Prove that $\sigma(A) = \sigma(B)$.

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Homework 6. Is the following matrix diagonalizable? If yes, diagonalize it.

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Homework 7. Consider the matrix from the previous exercise. Is it orthogonal? Hermitian? Symmetric? Skew-symmetric? Invertible?

Homework 8. Consider the following matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1) Compute A^4.
(2) Show that $I - A$ is invertible and compute its inverse.
(3) Compute the exponential of the matrix A.

Homework 9. Let $A \in \mathbb{K}^{n\times n}$ be an invertible matrix. Show that A^{-1} can be expressed as a polynomial of degree $\leq n-1$.

Homework 10. Let $A \in \mathbb{K}^{n\times n}$ and let $\langle \cdot, \cdot \rangle$ denotes the traditional Hermitian inner product. Show that

$$A^*A = AA^* \Leftrightarrow \langle A\mathbf{u}, A\mathbf{v}\rangle = \langle A^*\mathbf{u}, A^*\mathbf{v}\rangle, \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{K}^n.$$

Homework 11. Let $A \in \mathbb{K}^{n\times n}$ and let $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ denote respectively the Hermitian and skew-Hermitian parts of A (remember that the decomposition $A = S + H$ exists and is unique due to the direct sum mentioned in class).

Show that A is normal (i.e. $AA^* = A^*A$) if and only if S and H commute.

Homework 12. Let $S = \{A \in \mathbb{K}^{n\times n} : a_{i,j} = 0 \text{ for } i > j \text{ and } A^T = A\}$ and $T = \{A \in \mathbb{K}^{n\times n} : a_{i,j} = 0 \text{ for } i > j \text{ and } A^T = -A\}$. Characterize both S and T.

Homework 13. (1) Let A be a skew symmetric matrix in $\mathbb{K}^{n\times n}$. Show that for any $\mathbf{x} \in \mathbb{K}^n$, $\mathbf{x}$ and $A\mathbf{x}$ are orthogonal.
(2) Show the converse.

Homework 14. Let $C^2(I)$ be the set of twice continuously differentiable functions over some interval $I \subset \mathbb{R}$. Furthermore let ψ be defined as

$$\begin{array}{ccl} C^2(I) & \to & C^0(I) \\ f & \mapsto & \psi(f) = f'' + a(x)f' + b(x)f, \end{array}$$

for some continuous functions a and b.

(1) Show that ψ is linear.
(2) Deduce from this that the set of functions f satisfying $f'' + a(x)f' + b(x)f = 0$ is a vector space.

Homework 15. Let $A \in \mathbb{K}^{n\times n}$. Show that $\sigma(A) = \sigma(A^T)$ and that $\sigma(AA^T) = \sigma(A^TA)$.

Homework 16. Let A be the $n \times n$ tridiagonal matrix with a on the diagonal entries, b on the upper diagonal entries, and c on the lower diagonal, for some $a, b, c \in \mathbb{K}$. Compute $\det(A)$.

Homework 17. Let $S \subset \mathbb{R}^{m\times m}$ be the set of symmetric matrices and T the set of upper triangular matrices.
(1) Give a basis for S and compute $\dim(S)$
(2) Give a basis for T and compute $\dim(T)$.
(3) Conclude about $\dim(S \cap T)$.

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