EXERCISE SHEET 2: MATRIX ANALYSIS

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Homework 1. Assume A and B are two non-singular matrices. Prove that

$$adj(AB) = adj(B).adj(A)$$

Homework 2. Let $A \in \mathbb{R}^{n\times n}$, for $n \geq 2$. Prove that

$$adj(adj(A)) = (\det(A))^{n-2}\, A.$$

Homework 3. Let S be a subset of a vector space V.

(1) What can be said about $\dim(\text{span}(S))$?
(2) Let $V = C^\infty(\mathbb{R}, \mathbb{R})$ the set of infinitely differentiable functions. Is V finite or infinite dimensional?

Homework 4. Consider the following three basis of $V = \mathbb{R}_2[x]$:

- $S = (x \mapsto 1, x \mapsto x, x \mapsto x^2)$,
- $T = (x \mapsto 1, x \mapsto 1 + x, x \mapsto 1 + x^2)$,
- $U = (x \mapsto x^2 + 1, x \mapsto 1 + x, x \mapsto x + x^2)$,

Consider the following mapping:

$$M : \begin{array}{lcl} \mathbb{R}_2[x] & \to & \mathbb{R}_2[x] \\ p & \mapsto & p' + Xp' \end{array}$$

(1) Show that M is a linear transformation.
(2) Compute its matrix representations when looking at is using all the different basis (i.e. U for both input and output spaces, T for both input and output spaces, and then U).
(3) Show that all these matrices are similar to each other. What is the P matrix appearing in the equivalence
(4) Compute now the matrix of this linear transformation when the input and output basis are not the same. (i.e. 6 matrices in total). Show that for all of these matrices, there exists a pair of non-singular matrices P and Q such that $A = QBP^{-1}$ (where A is one of those matrices and B is another one). What do P and Q correspond to?

Homework 5. Let $A = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Fow which values of a, b, c is A invertible?

Homework 6. Let $(x_i)_{i=1}^n$ be n numbers in $\mathbb{R}$. Let A be the matrix such that $a_{i,j} = x_i^j$. Show that A is invertible $\Leftrightarrow x_i \neq x_j$ for $i \neq j$.

Homework 7. Let $A \in \mathbb{R}^{n\times n}$. Assume that for all $1 \leq i \leq n$, $\sum_{i=1}^n a_{i,j} = 1$. Prove that $\lambda = 1$ is an eigenvalue and give one of its eigenvectors.

Homework 8. Let $t \in \mathbb{R}$ and let $A = \begin{bmatrix} 1 & t & t & \cdots & t \\ t & 1 & t & \cdots & t \\ t & \vdots & \ddots & \cdots & t \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t & t & t & \cdots & 1 \end{bmatrix}$. Find the determinant, eigenvalues, eigenvectors of the matrix A and diagonalize it.

Homework 9. Let $A \in \mathbb{K}^{n\times n}$ and assume A is diagonalizable.

Date: November 28, 2018.

(1) Compute, using a power series, $exp(A)$. Verify, by analyzing the continuity of the partial sums, that this operation is well-defined!
(2) Does it hold that $\exp(A+B) = \exp(A)\exp(B)$. If yes, prove it, if no, give an example and a condition for the formula to be true.
(3) Assume moreover $A^3 = 0$. What is $\exp(A)$?
(4) What is $\exp(A^T)$?
(5) Assume $(\lambda, \mathbf{x})$ is an eigenpair of A. What can be said about the eigenvalues and / or eigenvectors of $\exp(A)$?

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