Exercise 1. Let U be a vector space and $V, W \subset U$ two subspaces. Are the following sets subspaces of U?

(1) $V \cap W := \{\mathbf{u} : \mathbf{u} \in V \text{ and } \mathbf{u} \in W\}$
(2) $V \cup W := \{\mathbf{u} : \mathbf{u} \in V \text{ or } \mathbf{u} \in W\}$
(3) $V + W := \{\mathbf{u} : \exists \mathbf{v} \in V, \mathbf{w} \in W : \mathbf{u} = \mathbf{v} + \mathbf{w}\}$

Exercise 2. Let $\mathbf{u}$ and $\mathbf{v}$ be two linearly independent vectors. Show that $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}\} = \text{span}\{\mathbf{u}, \mathbf{v}\} = \text{span}\{\mathbf{u}, \mathbf{u}+\mathbf{v}\}$.

Exercise 3. Write down the definition of what it means to be linearly dependent.

Exercise 4. Consider $V = \mathbb{R}_n[x]$. Are the following families linearly dependent?

- $(1, x, \cdots, x^n)$
- $(1, 1+x, 1+x+x^2, \cdots, 1+x+\cdots+x^{n-1}+x^n)$
- $(1, 1+x, 1+x^2, \cdots, 1+x^n)$
- $(1+x, x+x^2, x^2+x^3, \cdots, x^{n-1}+x^n, x^n+1)$

Exercise 5. Are the following families generating? Linearly independent? Basis?

- $(1, x, \cdots, x^n)$
- $(1, 1+x, 1+x+x^2, \cdots, 1+x+\cdots+x^{n-1}+x^n)$
- $(1, 1+x, 1+x^2, \cdots, 1+x^n)$
- $(1+x, x+x^2, x^2+x^3, \cdots, x^{n-1}+x^n, x^n+1)$

Exercise 6. What is the dimension of the following vector spaces:

- $\mathbb{R}_n[x]$
- $\mathbb{R}[x]$
- $\mathbb{R}^n$
- $\mathbb{C}^n$

Exercise 7. Let $C^1(\mathbb{R})$ be the set of continuously differentiable functions. Verify that $T : C^1 \to C^0, f \mapsto f'$ is a linear map.

Exercise 8. Prove that for any vector spaces V, W and any linear map $f : V \to W$, $f(0) = 0$.

Exercise 9. Let $f : U = \mathbb{R}_3[x] \to V = \mathbb{R}_3[x]$ be defined as the differentiation operator. Compute the matrices associated to f given the following basis

- $U = \text{span}(1, x, x^2, x^3)$ and $V = \text{span}(1, x, x^2, x^3)$.
- $U = \text{span}(1, x, x^2, x^3)$ and $V = \text{span}(1, 1+x, 1+x^2, 1+x^3)$.
- $U = \text{span}(1, 1+x, 1+x+x^2, 1+x+x^2+x^3)$ and $V = \text{span}(1, 1+x, 1+x+x^2, 1+x+x^2+x^3)$.

Exercise 10. Prove that the range and kernel of a linear mapping are indeed subspaces.

Exercise 11. Let $f : V \to W$, $S = (\mathbf{v}_1, \mathbf{v}_k)$ and $T = (f(\mathbf{v}_i))_i$. What can be said about T if

- S is a spanning set?
- S is linearly dependent?
- S is linearly independent?
- S is a basis?

Exercise 12. Show that the trace is linear and prove the following identity:

$$tr(AB) = tr(BA), \text{ for any } A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}.$$

Exercise 13. Prove or compute the following results:

- $\det(AB) = \det(A)\det(B)$
- Computations for 2×2 matrices and Sarrus' rule for 3×3.
- $\det(A^T) =?$
- $Aadj(A) = adj(A)A = \det(A)I$, where $adj(A)_{i,j} = (-1)^{i+j}A_{j,i}$ is the adjunct or adjugate matrix.

Exercise 14. Are all sets of these particular matrices subspaces of the vector space of matrices? In case of vector subspaces, what are their dimensions and give some basis.

Exercise 15. Which kind of structure does the set of symmetric matrices have?

Exercise 16. Prove that A is invertible if and only if $\det(A) \neq 0$ and give a formula for its inverse.

Exercise 17. Let T be an upper triangular matrix. Show that $\det(T) = \prod t_{ii}$.

Exercise 18. Let f be the differential operator on the set of degree 2 polynomials. Let $S = (1, x, x^2)$ and $T = (1, 1 + x, 1 + x + x^2)$. Furthermore, let A be the representation of f in the basis S and B the matrix representing f in T. Show that $A \sim B$. What does P represent?

Exercise 19. Let S be the set of symmetric matrices and T the set of skew-symmetric matrices. Show that $\mathbb{K}^{n\times n} = S \oplus T$.

Exercise 20. Verify that the eigenspaces are indeed vector spaces.

Exercise 21. Show that

$$p_A(x) = (-1)^n x^n + (-1)^{n-1} tr(A) + \cdots + \det(A)$$

and show that

$$tr(A) = \sum_{i=1}^{n} \lambda_i \quad \det(A) = \prod_{i=1}^{n} \lambda_i$$

where the λ_i are the n (possibly complex and repeated eigenvalues of A).

Conclude that A is invertible $\Leftrightarrow 0 \notin \sigma(A)$.

Exercise 22. Let A and B be two square matrices such that $A \sim B$. It holds

$$tr(A) = tr(B)$$
$$\det(A) = \det(B).$$

Exercise 23. Show the following: there exists a non-singular matrix V and a diagonal matrix D such that $A = VDV^{-1}$ if and only if there exists n linearly independent eigenvectors $\mathbf{v}_i$ with respective eigenvalues λ_i.

Homework 1. Show that $tr(A^*A) = 0 \Leftrightarrow A = 0$.

Homework 2. True or false (and justify) the following sets are multiplicative groups

- $G_1 = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$.
- $G_2 = \{A \in \mathbb{R}^{n\times n} : |\det(A)| = 1\}$.
- $G_3 = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0 \text{ and } a_{i,j} \in \mathbb{N}\}$.
- $G_4 = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0 \text{ and } a_{i,j} \in \mathbb{Z}\}$.

Homework 3. Given $A \in \mathbb{K}^{m\times n}$, prove or disprove the following statements (remember that A^* is the adjoing of A, i.e. its conjugate transpose – this is equivalent to the transpose of a matrix in case $\mathbb{K} = \mathbb{R}$).

(1) $rk(A^*) = rk(A)$.
(2) $\dim(ker(A)) = \dim(ker(A^*))$.
(3) $ker(A) = ker(A^*A)$.
(4) $\dim(ker(A)) = \dim(ker(A^*A))$.
(5) $rk(A) = rk(A^*A)$.

Homework 4. Prove that $AB = 0 \Leftrightarrow R(B) \subset ker(A)$.

Homework 5. (1) Let A and B be two square matrices. Assume that A is invertible. Show that $p_{AB}(x) = p_{BA}(x)$.
(2) Assume both A and B are singular. Show that $p_{AB}(x) = p_{BA}(x)$.
(3) What happens in case $A \in \mathbb{K}^{m\times n}$ and $B \in \mathbb{K}^{n\times m}$.

Homework 6. Let $A \in \mathbb{K}^{n\times n}$ and let p denotes any polynomial. Show that if $(\lambda, \mathbf{x})$ is an eigenpair of A then $(p(\lambda), \mathbf{x})$ is an eigenpair of $p(A)$. Use this to prove the theorem of Cayley-Hamilton in case of diagonalizable matrices: $p_A(A) = 0$. (You may also prove the theorem in a more general form, i.e. not necessarily when A is diagonalizable. In this case, you can use the fact that matrices over $\mathbb{C}$ which are diagonalizable form a dense subset of $\mathbb{C}^{n\times n}$ and then use this to conclude about matrices with entries in $\mathbb{R}$.)

Homework 7. (1) Let $A, B \in \mathbb{K}^{n\times n}$. Show that $AB = I \Leftrightarrow BA = I$.
(2) Show that this is no longer true in case $A \in \mathbb{K}^{m\times n}$ and $B \in \mathbb{K}^{n\times m}$.

Homework 8. Diagonalize (if possible) the following matrices and factor their characteristic polynomials in $\mathbb{R}$ and $\mathbb{C}$.

(1) $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$.

(2) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

(3) $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

(4) $A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$.

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