

Fusion Frames and Distributed Sparsity

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ABSTRACT. We analyze the problem of recovering signals from low quality sensing devices, and implement a combination of compressed sensing and distributed sensing to model so-called fusion frames structured signals. Within our framework, it is possible to recover signals with high accuracy by increasing the total number of measurements, while keeping the cheap sensors at hand. We show that, under suitable constraints, nonsparse signals can be recovered with high probability. Moreover, we show that, with the use of a few linear measurements, cheap sensors are sufficient when combined with the fusion frame methodology. Using our new method, we show that it is possible, and sometimes necessary, to split a signal via local projections for accurate, stable, and robust estimation.

1. Introduction

In applications of sampling theory, a good sensor network design ensures that recovery from sufficiently many samples of the signal of interest $\mathbf{x} \in \mathbb{K}^N$ is possible. Here \mathbb{K}^N denotes a finite dimensional Hilbert space over the field of real or complex numbers. When anticipating that the signal $\mathbf{x} \in \mathbb{K}^N$ is sparse, we can start with scarce, noisy measurements $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e} \in \mathbb{K}^m$, $m \ll N$, and recover a sparse $\hat{\mathbf{x}} \in \mathbb{K}^N \approx \mathbf{x}$. The recovery is a result of solving the mathematical program

$$(\ell^0\text{-min}) \quad \hat{\mathbf{x}} := \operatorname{argmin} \|\mathbf{z}\|_0, \quad \text{subject to } \|\mathbf{A}\mathbf{z} - \mathbf{y}\|_2 \leq \eta.$$

The $(\ell^0\text{-min})$ problem is NP-Hard, and its solution is approximated by the solution of its convex relaxation, known as the Basis Pursuit denoising

$$(\text{BPDN}) \quad \hat{\mathbf{x}} := \operatorname{argmin} \|\mathbf{z}\|_1, \quad \text{subject to } \|\mathbf{A}\mathbf{z} - \mathbf{y}\|_2 \leq \eta.$$

For a given *sparsity* s of \mathbf{x} , the number of random subgaussian linear measurements needs to grow as $m \gtrsim s \log(N/s)$ for $\hat{\mathbf{x}}$ to be a good enough approximation to \mathbf{x} , which means, we have a stable and robust recovery, provided we needed but relatively few measurements.

We analyze in this paper the problem of signal sampling and recovery when the quality of sensors is constrained, hereby limiting the number of measurements m , independently of the sparsity or complexity of the signal of interest. The sensors constraints can be due to many reasons such as cost (e.g. using sensors at a coarser resolution is cheaper than one at the finest), legal regulation (for instance in nuclear medicine where one must not expose a patient to high amounts of radiations at

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once). The quality of recovery is not ensured when the signals being sampled are *not sparse enough*, given the constraints of the sensors used.

In this paper, we assume certain constraints on the sensor design which are fixed due to some outside reasons. Under these assumptions, we take on the following challenge: split the information carried by the signal in a clever way so that a mathematical recovery is possible; that is, we apply the fusion frame theory to the signal recovery problem. We show that it is possible to handle very complex signals in an efficient and stable manner.

1.1. Motivation: Unavailability of high quality observation devices.

A time-invariant bounded linear operator is represented by a circulant matrix \mathbf{A} . Suppose \mathbf{A} represents a sensing device whose number of rows m is physically limited by the sampling rate (or resolution) of the device. Suppose also that the sparsity of the sampled signal \mathbf{x} is substantially larger than what the standard compressed sensing techniques could handle when working with \mathbf{Ax} .

In this context, the limitations on the sensing devices combined with the (potentially) high number of non-zeros in the signals makes it impossible for a state-of-the-art algorithm to recover the unknown signal \mathbf{x} accurately. As illustrated in Figure 1, we suggest to apply n such devices in parallel after pre-filtering. The fused compressed sensing technique introduced here allows to resolve the problem that otherwise a single device can not.

We present an example of such a real-life scenario to elaborate on the necessity of the proposed fused compressed sensing technique: Suppose an application

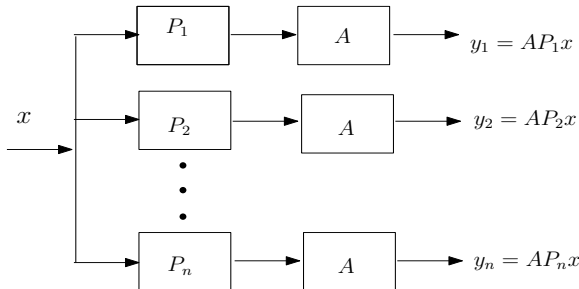


FIGURE 1. Sparse linear array geometry

requires a sophisticated sensing device $\tilde{\mathbf{A}}$ of very high sensing rate. Such a device may be very expensive, or not available on the market, which is why we combine 10 parallel projections or filtering operations $\{P_j\}_{j=1}^{10}$ prior to measuring using 10 lower rate devices, each with measurement matrix \mathbf{A} . The (sparse) signal \mathbf{x} is then subsequently recovered by various techniques in each subspace/channel and, through the theory of fusion frames [12, 13, 10], merged into a single vector. Such a fusion operation is possible if $\{P_j\}_{j=1}^{10}$ are the projections or any filtering operations satisfying $\mathbf{AI} \leq \sum_i P_j^* P_j \leq \mathbf{BI}$ for some $0 < A \leq B < \infty$.

Here is why such a pre-filtering/projection mechanism manages to distribute the sparsity. Since the signal has a 20GHz bandwidth, the non-zeros can be assumed to be distributed within the entire bandwidth. Otherwise, the bandwidth could be assumed to be smaller.

In order for the fusion operator to be bounded away from 0, the projections P_i are certainly bandpass in nature. Namely, P_i 's are distributed over adjacent and overlapping bands so that there won't be "holes" in the frequency domain. Consequently, each P_i represents different bandpass filtering with different bands and carries a "different part of sparsity"; that is, the sparsity is distributed in different bands.

Evidently, if we design such a fused compressed sensing technique, by enabling the subdivision of sparsity of \mathbf{x} into individual subspaces, the sparse recovery problem becomes a feasible one and can be resolved by multiple sensing devices with sampling rates lower than 2GHz, which are not only widely available but also economical.

The value of fused compressed sensing techniques presented in this paper is clearly reflected in this situation where an otherwise too expensive or impossible problem can now be resolved by using a number of lower resolution/sampling rate devices and by making a *reasonable* number of observations, and processed by the fusion frame theory.

1.2. Contributions. In this paper we combine mathematical tools from compressed sensing and fusion frame theory to break the limitations on the signal complexity induced by traditional recovery methods. Our Theorem 3.11 states the following: Given a fusion frame system $\mathcal{W} = (W_i, P_i)_{i=1}^n$ for \mathbb{K}^N with frame operator S and frame bounds $0 < C \leq D < \infty$, and a matrix $\mathbf{A} \in \mathbb{K}^{m \times N}$ satisfying a Partial RIP condition (see Definition 3.4), we can recover a \mathbf{s} distributed sparse vector $\mathbf{x} \in \mathbb{K}^N$ (see Definition 3.1) as

$$\widehat{\mathbf{x}} = S^{-1} \sum_{i=1}^n \widehat{\mathbf{x}}^{(i)},$$

where the $\widehat{\mathbf{x}}^{(i)}$ are obtained as solutions to the n local s -sparse recovery problems

$$\widehat{\mathbf{x}}^{(i)} := \operatorname{argmin} \|\mathbf{z}\|_0, \quad \text{subject to } \|\mathbf{A}P_i\mathbf{x} - \mathbf{y}^{(i)}\|_2 \leq \eta.$$

Moreover, assuming certain uniformity over all subspaces, the error of solution approximation $\|\widehat{\mathbf{x}} - \mathbf{x}\|_2$ is well-controlled by inequality (3.5).

In Section 2 we review the basic tools from compressed sensing and fusion frame theory. We extend these tools to the case of recovery with assumed local redundancy in the fusion frame decomposition in Section 3. In particular, we derive a mathematical theory allowing to work with fusion frames where sparsity is exploited along a subspace decomposition. Our theoretical results are illustrated on numerical examples where we try to recover a signal with dense Fourier spectrum from cheap sensors in Section 4.

2. General tools and models

2.1. Compressed sensing tools. Compressed sensing (CS) relies ([19, 17] and references within) on the inherent sparsity of natural signals \mathbf{x} for their recovery from seemingly few measurements $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ for some linear measurement matrix $\mathbf{A} \in \mathbb{K}^{m \times N}$, with $m \ll N$. Here the vector $\mathbf{e} \in \mathbb{K}^m$ represents the measurement noise, and is assumed to be bounded, $\|\mathbf{e}\|_2 \leq \eta$. With the sparsity assumption, CS aims at finding (approximate) solutions to the (ℓ^0 -min) problem. General recovery

guarantees ensure that the recovery is stable and robust, that is, the solution $\hat{\mathbf{x}}$ satisfies

$$\begin{aligned}\|\hat{\mathbf{x}} - \mathbf{x}\|_2 &\leq \frac{C}{\sqrt{s}} \sigma_s(\mathbf{x})_1 + D\eta, \\ \|\hat{\mathbf{x}} - \mathbf{x}\|_1 &\leq C\sigma_s(\mathbf{x})_1 + D\sqrt{s}\eta,\end{aligned}$$

where $\sigma_s(\mathbf{x})_1 := \min_{\mathbf{z}: \|\mathbf{z}\|_0 \leq s} \|\mathbf{x} - \mathbf{z}\|_1$ is called the best s -term approximation of \mathbf{x} .

Plethora of conditions (some of which we look closer in Section 3) on \mathbf{A} have been derived to ensure that the previous (or similar) estimates hold. These are based on restricted isometry constants ($\delta_{2s} < \sqrt{2} - 1$ [11] or $\delta_{2s} < 4/\sqrt{41}$ [19], $\delta_s < 1/3$), null space properties (\mathbf{A} fulfills the robust and stable NSP(s, ρ, τ) if $\|\mathbf{v}_S\|_1 \leq \rho\|\mathbf{v}_{\bar{S}}\|_1 + \tau\|\mathbf{A}\mathbf{v}\|_2$ for any vector $\mathbf{v} \in \mathbb{K}^N$, and S any index set with $|S| \leq s$), coherence ($\mu_1(s) + \mu_1(s-1) < 1$), quotient property, and so on. Similar conditions and bounds can be found when greedy and thresholding algorithms are used to approximate (ℓ^0 -min) (see [5, 18, 7, 6, 23, 21] for some relatively recent results in this direction).

There is still a need for building adequate sensing matrices \mathbf{A} , which ensure good recovery bounds. So far, no deterministic matrices \mathbf{A} can be built with reasonably few rows, and we have to rely on randomness to build measurement matrices. It has been shown, for instance, that matrices with independent random subgaussian entries fulfill the RIP(s, δ) under certain constraints on the number of measurements [4, 22].

2.2. Frames, fusion frames, and distributed signal processing. By definition, a sequence $\mathcal{F} = \{\mathbf{f}_i\}_{i \in I}$ in a Hilbert space \mathcal{H} is a frame [14] for \mathcal{H} if there exist $0 < A \leq B < \infty$ (lower and upper frame bounds) such that

$$A\|\mathbf{f}\|^2 \leq \sum_{i \in I} |\langle \mathbf{f}, \mathbf{f}_i \rangle|^2 \leq B\|\mathbf{f}\|^2 \text{ for all } \mathbf{f} \in \mathcal{H}.$$

The representation space associated with \mathcal{F} is $\ell^2(I)$ and its analysis and synthesis operators are respectively given by $T(\mathbf{f}) = \{\langle \mathbf{f}, \mathbf{f}_i \rangle\}_{i \in I}$ and $T^*(\{c_i\}_{i \in I}) = \sum_{i \in I} c_i \mathbf{f}_i$, for all $\mathbf{f} \in \mathcal{H}$ and $\{c_i\}_{i \in I} \in \ell^2(I)$. It shows that the frame operator $S := T^*T$ is a positive, self-adjoint and invertible operator; this means that recovery of any $\mathbf{f} \in \mathcal{H}$ is possible, if $S\mathbf{f}$ is known; however, computing S^{-1} can be computationally challenging. Each frame \mathcal{F} is accompanied by at least one so-called dual frame $\mathcal{G} = \{\mathbf{g}_i\}_{i \in I}$, which satisfies

$$\mathbf{f} = \sum_{i \in I} \langle \mathbf{f}, \mathbf{f}_i \rangle \mathbf{g}_i = \sum_{i \in I} \langle \mathbf{f}, \mathbf{g}_i \rangle \mathbf{f}_i \text{ for all } \mathbf{f} \in \mathcal{H}.$$

Whenever a frame is A -tight ($A = B$), the problem of function reconstruction is simplified, since in this case the frame operator is given by $S = A\mathbf{I}$, with \mathbf{I} denoting the identity operator.

Fusion frames have been initially created [12, 13] to model the setting of a wireless sensor network. A large sensor network is typically split into redundant sub-networks; the local measurements within each sub-network are sent to a local sub-station, which submits the gathered information further to a central processing station for final reconstruction. Each sub-network is related to a frame for a subspace of a Hilbert space. The subspaces have to satisfy certain overlapping

properties. The theory of fusion frames derives a number of means to fuse together sub-station/subspace information for a complete signal reconstruction.

DEFINITION 2.1 (Fusion frames). Given an index set I , let $\mathcal{W} := \{W_i \mid i \in I\}$ be a family of closed subspaces in \mathcal{H} . We denote by P_i the orthogonal projections onto W_i . Then \mathcal{W} is a fusion frame, if there exist $C, D > 0$ such that

$$C\|\mathbf{f}\|^2 \leq \sum_{i \in I} \|P_i(\mathbf{f})\|^2 \leq D\|\mathbf{f}\|^2 \text{ for all } \mathbf{f} \in \mathcal{H}.$$

REMARK 2.2. Fusion frames are often accompanied by respective weights. In the weighted case, the frame condition reads $C\|\mathbf{f}\|^2 \leq \sum_{i \in I} v_i^2 \|P_i(\mathbf{f})\|^2 \leq D\|\mathbf{f}\|^2$ for all $\mathbf{f} \in \mathcal{H}$ and some positive weights $(v_i)_{i \in I}$. All (unweighted) results derived in this paper apply mutatis mutandis to the weighted case.

The following theorem [12] illustrates the relationship between the local and global properties of a fusion frame:

THEOREM 2.3. For each $i \in I$, let W_i be a closed subspace of \mathcal{H} , and let $\mathcal{F}_i = \{\mathbf{f}_{ij} \mid j \in J_i\}$ be a frame for W_i , with frame bounds A_i, B_i . If $0 < A = \inf_{i \in I} A_i \leq \sup_{i \in I} B_i = B < \infty$, then the following conditions are equivalent:

- $\cup_{i \in I} \{\mathbf{f}_{ij} \mid j \in J_i\}$ is a frame for \mathcal{H} .
- $\{(W_i, \mathcal{F}_i)\}_{i \in I}$ is a fusion frame for \mathcal{H} .

In particular, if $\{(W_i, \mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for \mathcal{H} with frame bounds C and D , then $\cup_{i \in I} \{\mathbf{f}_{ij} \mid j \in J_i\}$ is a frame for \mathcal{H} with frame bounds AC and BD .

In fusion frame theory, an input signal is represented by a collection of vector coefficients that represent the projection onto each subspace. The representation space used in this setting is

$$\left(\sum_{i \in I} \oplus W_i \right)_{\ell^2} = \{ \{\mathbf{f}_i\}_{i \in I} \mid \mathbf{f}_i \in W_i \}$$

with $\{\|\mathbf{f}_i\|\}_{i \in I} \in \ell^2(I)$. The analysis operator is given by $T(\mathbf{f}) := \{P_i(\mathbf{f})\}_{i \in I}$ for all $\mathbf{f} \in \mathcal{H}$, while its adjoint is the synthesis operator $T^* : (\sum_{i \in I} \oplus W_i)_{\ell^2} \rightarrow \mathcal{H}$, defined by

$$T^*(\mathbf{f}) = \sum_i \mathbf{f}_i, \text{ where } \mathbf{f} = \{\mathbf{f}_i\}_{i \in I} \in \left(\sum_{i \in I} \oplus W_i \right)_{\ell^2}.$$

The fusion frame operator $S = T^*T$ is given by $S(\mathbf{f}) = \sum_{i \in I} P_i(\mathbf{f})$. It is easy to verify that S is a positive and invertible operator on \mathcal{H} . A distributed fusion processing is feasible in an elegant way, since the reconstruction formula for all $\mathbf{f} \in \mathcal{H}$ is

$$\mathbf{f} = \sum_{i \in I} S^{-1} P_i(\mathbf{f}) = S^{-1} \sum_{i \in I} P_i(\mathbf{f}).$$

2.3. Signal recovery in fusion frames. This work describes an approach for sensing and reconstructing signals in a fusion frame structure. As presented above, given some local information $\mathbf{x}^{(i)} := P_i(\mathbf{x})$, for $1 \leq i \leq n$, a vector can easily be reconstructed by applying the inverse fusion frame operator

$$(2.1) \quad \mathbf{x} := S^{-1}S(\mathbf{x}) = S^{-1} \sum_{i=1}^n P_i(\mathbf{x}) = S^{-1} \left(\sum_{i=1}^n \mathbf{x}^{(i)} \right).$$

Throughout this work, we assume that the projected vectors are sampled independently from one another, with n devices modeled by the same sensing matrix \mathbf{A} . The task at hand is as follows: *Use the measurements $\mathbf{y}^{(i)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{e}^{(i)}$ to compute an estimate $\widehat{\mathbf{x}}^{(i)}$ of the local component $\mathbf{x}^{(i)}$; then compute the approximation $\widehat{\mathbf{x}}$ of the signal \mathbf{x} via (2.1).*

From this point on, there are two ways of thinking about the problem. In a first scenario, the signals are measured in the subspace and the recovery of each $\mathbf{x}^{(i)}$ is done locally. In other words, it accounts for solving n (ℓ^0 -min) problems (or their approximations via (BPDN) for instance) in the subspaces, and then transmitting the estimated local signals to a central unit taking care of the fusion via an analogue of (2.1). The other approach consists of transmitting the local observations $\mathbf{y}^{(i)} = \mathbf{A}P_i\mathbf{x} + \mathbf{e}^{(i)}$ to a central processing unit which takes care of the whole reconstruction process. In this case, the vector \mathbf{x} can be recovered by solving a unique (ℓ^0 -min) problem directly with, letting \mathbf{I}_n denote the n dimensional identity matrix,

$$(2.2) \quad \mathbf{y} = \begin{pmatrix} \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(n)} \end{pmatrix} = \mathbf{I}_n \otimes \mathbf{A} \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} \mathbf{x}$$

While the latter case is interesting on its own (see for instance [16]), we investigate here some results for the first case. Our results can be generalized further, integrating ideas where the measurements matrix (here, the sensors) vary locally, as is the case in [16] or driven with some structured acquisition (see for instance [9]).

We would like to put our work in context. This paper is not the first one to describe the use of fusion frames in sparse signal recovery. However, it is inherently different from previous works in [8, 2]. In [8] the authors provide a framework for recovering sparse fusion frame coefficients. In other words, given a fusion frame system $\{(W_i, P_i)\}_{i=1}^n$ a vector is represented on this fusion frame as a set of n vectors $(\mathbf{x}^{(i)})_{i=1}^n$ where each of the $\mathbf{x}^{(i)}$ corresponds to the coefficient vector in subspace W_i . The main idea of the authors is that the original signal may only lie in few of the n subspaces, implying that most of the $\mathbf{x}^{(i)}$ should be 0. To rephrase the problem, we can say that the vector has to lie in a sparse subset of the original fusion frame. In terms of the band-pass filtering discussed in the introduction, this would be equivalent to the coefficients to 0 within a whole band. While [8] is concerned with some recovery guarantees under (fusion-type) RIP and average case analysis, the paper [2] gives uniform results for subgaussian measurements and derives results on the minimum number of vector measurements required for robust and stable recovery.

3. An extension of traditional compressed sensing

Our goal is to ensure robust and stable recovery of complex signals by smartly combining local information. For this purpose, we extend the tools of traditional CS via the use of fusion frames. We note that the authors of [16, 15] introduce a model which corresponds to the idea of solving a single sparse recovery problem. They analyze RIP conditions to ensure stable and robust recovery of \mathbf{x} via a single matrix similar to Equation (2.2).

We start with a fusion frame $\mathcal{W} = (W_i, P_i)_{i=1}^n$ for \mathbb{K}^N , and assume that our signal of interest is $\mathbf{x} \in \mathbb{K}^N$. We aim at signal recovery by means of (a) CS in

local subspaces and (b) fusion processing; that is, the local pieces of information are computed as solutions to the problems

$$(\mathcal{P}_{1,\eta}) \quad \min_{\mathbf{z} \in \mathbb{K}^N} \|\mathbf{z}\|_1 \quad \text{s.t.} \quad \|\mathbf{A}P_i\mathbf{z} - \mathbf{y}^{(i)}\|_2 \leq \eta,$$

where P_i is the projection operator related to subspace W_i . In the noiseless case, the problem is solved by the basis pursuit

$$(\mathcal{P}_{1,0}) \quad \min_{\mathbf{z} \in \mathbb{K}^N} \|\mathbf{z}\|_1 \quad \text{s.t.} \quad \mathbf{A}P_i\mathbf{z} = \mathbf{y}^{(i)}.$$

Numerically, solving a set of smaller subproblems is typically more efficient than solving a single large system. In addition, the compressed sensing step in local subspaces can be done in a parallel fashion, greatly improving the computational efficiency.

DEFINITION 3.1. Let $s > 0$ and set $\mathbf{s} = (s, \dots, s)$ be the n -tuple with identical entries s . A signal $\mathbf{x} \in \mathbb{K}^N$ is said to be \mathbf{s} -uniformly distributed sparse (or \mathbf{s} -UD sparse) with respect to a fusion frame $\mathcal{W} = (W_i, P_i)_{i=1}^n$, if $\|P_i(\mathbf{x})\|_0 \leq s$, for every $1 \leq i \leq n$. We also refer to \mathbf{s} as a *uniform sparsity pattern* of \mathbf{x} with respect to \mathcal{W} .

REMARK 3.2. It is important to understand what it means for a signal to be distributed sparse. In particular, an s sparse signal need not be distributed \mathbf{s} sparse. Conversely, an \mathbf{s} distributed sparse signal might also be dense.

An important particular case, is when dealing with projections parallel to the original axis of the space. In this case, an s sparse vector is also \mathbf{s} distributed sparse, where the worst case scenario is if all the s non zero terms fall within a single subspace. An optimal design of the fusion frame is dependent on the signal, and plays a great role in our approach.

By $\Sigma_{\mathbf{s}}^{(\mathcal{W})}$ we denote the set of all \mathbf{s} -UD sparse vectors with respect to the family of subspaces \mathcal{W} . Since the sparsity of the signal is uniformly distributed among the subspaces, the usual CS recovery guarantees ensure that

$$m \asymp s \log(N/s)$$

(subgaussian) observations per subspace are required for a stable and robust recovery of \mathbf{x} . This accounts for a total number of measurements scaling as $m_T \asymp ns \log(N/s)$. Most of the computations can be easily carried in a distributed setting, where only pieces of the information are available. Only the fusion process requires all the local information to compute the final estimation of a signal.

Note that, for $p > 0$ and a UD sparsity pattern $\mathbf{s} = (s, \dots, s)$ with $s \in \mathbb{N}^+$, the ℓ^p errors of best \mathbf{s} -term approximations are understood in a distributed setting, and are embedded in the vector

$$\sigma_{\mathbf{s}}^{\mathcal{W}}(\mathbf{x})_p := (\sigma_s(P_1\mathbf{x})_p, \sigma_s(P_2\mathbf{x})_p, \dots, \sigma_s(P_n\mathbf{x})_p)^T.$$

3.1. Partial properties. The *null space property* (NSP) is used in the CS literature as a necessary and sufficient condition for the sparse recovery problem via (ℓ^0 -min). A matrix \mathbf{A} is said to satisfy the (robust) null space property with parameters $\rho \in (0, 1)$ and $\tau > 0$ relative to a set $S \subset \{1, \dots, N\}$ if

$$\|\mathbf{v}_S\|_1 \leq \rho \|\mathbf{v}_{\bar{S}}\|_1 + \tau \|\mathbf{A}\mathbf{v}\|_2, \quad \text{for all } \mathbf{v} \in \mathbb{K}^N.$$

More generally, we say that the matrix \mathbf{A} satisfies the NSP of order s if it satisfies the NSP relative to all sets S such that $|S| \leq s$. Here we talk about a sparsity

pattern and ask that the NSP property be valid for all local subspaces up to a certain (local) sparsity level.

DEFINITION 3.3 (Uniformly distributed partial null space property (UDP-NSP)). Let $\mathcal{W} = (W_i, P_i)_{i=1}^n$ be a fusion frame for \mathbb{K}^N , and let \mathbf{s} be a UD sparsity pattern (with entries s). A sensing matrix $\mathbf{A} \in \mathbb{K}^{m \times N}$ is said to fulfill the UDP-NSP with UD pattern \mathbf{s} with respect to \mathcal{W} and uniform constants $\rho \in (0, 1)$ and $\tau > 0$ if

$$\|(P_i \mathbf{v})_{S_i}\|_1 \leq \rho \|(P_i \mathbf{v})_{\overline{S_i}}\|_1 + \tau \|\mathbf{A} \mathbf{v}\|_2,$$

for all $\mathbf{v} \in \mathbb{K}^N$, $1 \leq i \leq n$, $S_i \subset W_i$, and $|S_i| \leq s$.

Definition 3.3 is reminiscent of the work on sparse recovery with partially known support [3]. Note that in our case, there is no need to enforce a condition on the vector \mathbf{v} to lie in the range of the other subspaces. In a sense, this is taken care of by the fusion process and the fact that we have multiple measurement vectors.

In CS, a stronger condition is often used to ensure recovery: the *Restricted Isometry Property* (RIP). A matrix is said to satisfy the $RIP(s, \delta)$ if it behaves like an isometry (up to a constant δ) on every s -sparse vector $\mathbf{v} \in \Sigma_s$. Formally speaking, $\mathbf{A} \in \mathbb{K}^{m \times N}$ satisfies $RIP(s, \delta)$, for some $s \geq 2$ and $\delta \in (0, 1)$ if

$$(3.1) \quad (1 - \delta) \|\mathbf{v}\|_2^2 \leq \|\mathbf{A} \mathbf{v}\|_2^2 \leq (1 + \delta) \|\mathbf{v}\|_2^2, \quad \text{for every } \mathbf{v} \in \Sigma_s.$$

The lowest δ satisfying the inequalities is called the *restricted isometry constant*. We want to derive similar properties on our sensing matrix for the distributed sparse signal model.

DEFINITION 3.4 (Uniform Partial RIP (UP-RIP)). Let $\mathcal{W} = (W_i, P_i)_{i=1}^n$ be a fusion frame, and let $\mathbf{A} \in \mathbb{K}^{m \times N}$. Assume that $\mathbf{A} P_i$ satisfies the $RIP(s, \delta)$ on each W_i , with $\delta \in (0, 1)$, $i \in I = \{1, \dots, n\}$. Then, we say that $\mathbf{A} \in \mathbb{K}^{m \times N}$ satisfies the UP-RIP with respect to \mathcal{W} , with uniform bound δ and uniform sparsity pattern \mathbf{s} .

REMARK 3.5. Definition 3.4 reduces to the definition of the classical RIP when $n = 1$ (one projection, one subspace).

The UP-RIP is characterized by an inequality similar to Equation (3.1):

PROPOSITION 1. Let $\mathcal{W} = (W_i)_{i=1}^n$ be a fusion frame for \mathbb{K}^N , with frame bounds $0 < C \leq D < \infty$. Let $\mathbf{A} \in \mathbb{K}^{m \times N}$ satisfy the UP-RIP with respect to \mathcal{W} , with uniform bound δ and sparsity pattern $\mathbf{s} = (s, \dots, s)$, and let $C_o = C(1 - \delta)$, $D_o = D(1 + \delta)$. Then, for any $\mathbf{v} \in \Sigma_s^{\mathcal{W}}$.

$$C_o \|\mathbf{v}\|_2^2 \leq \sum_{i=1}^n \|\mathbf{A} \mathbf{v}_i\|_2^2 \leq D_o \|\mathbf{v}\|_2^2.$$

PROOF. Using the fusion frame inequality, and inequalities (3.1) on each W_i , $1 \leq i \leq n$, we obtain

$$\begin{aligned} C_o \|\mathbf{v}\|_2^2 &\leq (1 - \delta) \sum_i \|\mathbf{v}_i\|_2^2 \leq \sum_i (1 - \delta) \|\mathbf{v}_i\|_2^2 \leq \sum_i \|\mathbf{A} \mathbf{v}_i\|_2^2 \\ &\leq \sum_i (1 + \delta) \|\mathbf{v}_i\|_2^2 \leq (1 + \delta) \sum_i \|\mathbf{v}_i\|_2^2 \leq D_o \|\mathbf{v}\|_2^2. \end{aligned}$$

□

We recall a standard RIP result for subgaussian matrices (Thm9.2 in [19]):

THEOREM 3.6. *Let $\varepsilon > 0$. Let \mathbf{A} be an $m \times N$ subgaussian random matrix. Then there exists a constant $C > 0$ (depending only on subgaussian parameters β, k) such that the RIP constant δ_s of $\frac{1}{\sqrt{m}}\mathbf{A}$ satisfies $\delta_s \leq \delta$ with probability at least $1 - \varepsilon$, if*

$$m \geq C\delta^{-2} (s \ln(eN/s) + \ln(2\varepsilon^{-1})).$$

It is relatively easy to show that:

THEOREM 3.7. *Let $\varepsilon > 0$. Let $\mathcal{W} = (W_i, P_i)_{i=1}^n$ be a fusion frame for \mathbb{K}^N . Let $\mathbf{A} \in \mathbb{K}^{m \times N}$ be a subgaussian matrix with parameters β, k . Then, there exists a constant $C = C_{\beta, k}$ such that the UP-RIP constants of $\frac{1}{\sqrt{m}}\mathbf{A}$ satisfy $\delta_s \leq \delta$, with probability at least $1 - \varepsilon$, provided*

$$m \geq C \frac{1}{\delta^2} (s \ln(eN/s) + \ln(2\varepsilon^{-1}n)).$$

REMARK 3.8. All Gaussian and Bernoulli random matrices are subgaussian random matrices, so Theorem 3.7 holds true for Gaussian and Bernoulli random matrices.

3.2. Recovery in general fusion frames settings. Using the tools introduced in Subsection 3.1, we show that any signals with uniform sparsity pattern \mathbf{s} can be recovered in a stable and robust manner via a fusion frame approach.

THEOREM 3.9. *Let $\mathbf{A} \in \mathbb{K}^{m \times N}$ and $\mathcal{W} = (W_i, P_i)_{i=1}^n$ a fusion frame with frame bounds $0 < C \leq D < \infty$ and frame operator S . Let $(\mathbf{y}^{(i)})_{i=1}^n$ be the linear measurements $\mathbf{y}^{(i)} = \mathbf{A}P_i\mathbf{x} + \mathbf{e}^{(i)}$, $1 \leq i \leq n$ for some uniformly bounded noise vectors $\mathbf{e}^{(i)}$ such that $\|\mathbf{e}^{(i)}\|_2 \leq \eta$. Denote by $\mathbf{x}^{(i)}$ the solution to the local Basis pursuit problems $(\mathcal{P}_{1, \eta})$.*

If the matrix \mathbf{A} satisfies the UDP-NSP with sparsity pattern \mathbf{s} with constants $0 < \rho < 1$ and $\tau > 0$ with respect to \mathcal{W} , then $\widehat{\mathbf{x}} = S^{-1} \sum_i \widehat{\mathbf{x}}^{(i)}$ approximates \mathbf{x} in the following sense:

$$(3.2) \quad \|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \leq \|\widehat{\mathbf{x}} - \mathbf{x}\|_1 \leq \frac{2}{C} \left(\frac{1 + \rho}{1 - \rho} \sum_{i=1}^n \sigma_{\mathbf{s}}(P_i\mathbf{x}) + \frac{2n\tau\eta}{1 - \rho} \right).$$

PROOF. The solution is given by the fusion process $\widehat{\mathbf{x}} = S^{-1} \left(\sum_{i=1}^n \widehat{\mathbf{x}}^{(i)} \right)$ with $\widehat{\mathbf{x}}^{(i)}$ the solutions to the local problems $(\mathcal{P}_{1, \eta})$. It holds

$$\|\mathbf{x} - \widehat{\mathbf{x}}\|_2 = \left\| S^{-1} \left(\sum_{i=1}^n P_i\mathbf{x} - \sum_{i=1}^n \widehat{\mathbf{x}}^{(i)} \right) \right\|_2 \leq C^{-1} \sum_{i=1}^n \left\| P_i\mathbf{x} - \widehat{\mathbf{x}}^{(i)} \right\|_2.$$

Thus

$$\|\mathbf{x} - \widehat{\mathbf{x}}\|_2 \leq C^{-1} \sum_{i=1}^n \left\| P_i\mathbf{x} - \widehat{\mathbf{x}}^{(i)} \right\|_1.$$

For each $i \in \{1, \dots, n\}$, we estimate the error on subspace W_i in the ℓ^1 sense: if $\mathbf{v} := P_i\mathbf{x} - \widehat{\mathbf{x}}^{(i)}$ and $S_i \subset W_i$ the set of best s_i components of \mathbf{x} supported on W_i , then the robust null space property yields

$$\|(P_i\mathbf{v})_{S_i}\|_1 \leq \rho \|(P_i\mathbf{v})_{\overline{S_i}}\|_1 + \tau \|\mathbf{A}\mathbf{v}\|_2.$$

Combining with [19, Lemma 4.15] stating

$$\|(P_i \mathbf{v})_{\overline{S_i}}\|_1 \leq \|P_i \widehat{\mathbf{x}^{(i)}}\|_1 - \|P_i \mathbf{x}\|_1 + \|(P_i \mathbf{v})_{S_i}\|_1 + 2 \|(P_i \mathbf{x})_{\overline{S_i}}\|_1.$$

Summing both inequalities, we arrive at

$$(1 - \rho) \|(P_i \mathbf{v})_{\overline{S_i}}\|_1 \leq \|P_i \widehat{\mathbf{x}^{(i)}}\|_1 - \|P_i \mathbf{x}\|_1 + 2 \|(P_i \mathbf{x})_{\overline{S_i}}\|_1 + \tau \|\mathbf{A} \mathbf{v}\|_2.$$

Applying once again the local robust NSP, it holds

$$\begin{aligned} \|P_i \mathbf{v}\|_1 &= \|(P_i \mathbf{v})_{S_i}\|_1 + \|(P_i \mathbf{v})_{\overline{S_i}}\|_1 \leq \rho \|(P_i \mathbf{v})_{\overline{S_i}}\|_1 + \tau \|\mathbf{A} \mathbf{v}\|_2 + \|(P_i \mathbf{v})_{\overline{S_i}}\|_1 \\ &\leq (1 + \rho) \|(P_i \mathbf{v})_{\overline{S_i}}\|_1 + \tau \|\mathbf{A} \mathbf{v}\|_2 \\ &\leq \frac{1 + \rho}{1 - \rho} \left(\|P_i \widehat{\mathbf{x}^{(i)}}\|_1 - \|P_i \mathbf{x}\|_1 + 2 \|(P_i \mathbf{x})_{\overline{S_i}}\|_1 \right) + \frac{4\tau}{1 - \rho} \|\mathbf{A} \mathbf{v}\|_2. \end{aligned}$$

Noticing that $P_i \widehat{\mathbf{x}^{(i)}} = \widehat{\mathbf{x}^{(i)}}$, and $\widehat{\mathbf{x}^{(i)}}$ being the optimal solution to $(\mathcal{P}_{1,\eta})$, it is clear that $\|\widehat{\mathbf{x}^{(i)}}\|_1 \leq \|P_i \mathbf{x}\|_1$ from what we can conclude that

$$(3.3) \quad \|P_i \mathbf{x} - \widehat{\mathbf{x}^{(i)}}\|_1 = \|P_i \mathbf{v}\|_1 \leq 2 \frac{1 + \rho}{1 - \rho} \sigma_s^{\mathcal{W}}(\mathbf{x})_{1,i} + \frac{4\tau}{1 - \rho} \|\mathbf{A} \mathbf{v}\|_2.$$

Remembering that $\|\cdot\|_2 \leq \|\cdot\|_1$ and summing up the contributions for all i in $\{1, \dots, n\}$ and applying the inverse frame operator finishes the proof. \square

3.2.1. UP-RIP based recovery. We will now demonstrate that the UP-RIP is sufficient for stable and robust recovery; the following result, showing the existence of random matrices satisfying the UDP-NSP, is crucial to our argument.

THEOREM 3.10. *Let $\mathbf{A} \in \mathbb{K}^{m \times N}$ be a matrix satisfying the UP-RIP($2\mathbf{s}, \delta$), with $\mathbf{s} = (s, \dots, s)$ and $\delta < 4/\sqrt{41}$ on all W_i , $1 \leq i \leq n$. Then, \mathbf{A} satisfies the UDP-NSP with constants*

$$(3.4) \quad \begin{aligned} \rho &:= \frac{\delta}{\sqrt{1-\delta^2}-\delta/4} < 1 \\ \tau &:= \frac{\sqrt{1+\delta}\sqrt{s}}{\sqrt{1-\delta^2}-\delta/4}. \end{aligned}$$

PROOF. The proof of this result consists in simply applying [19, Theorem 6.13] to every subspace independently. \square

By Theorem 3.9 and Theorem 3.10, we have a first recovery result under UP-RIP:

THEOREM 3.11. *Let $\mathcal{W} = (W_i, P_i)_{i=1}^n$ be a fusion frame for \mathbb{K}^N with frame operator S and frame bounds $0 < C \leq D < \infty$. Let $\mathbf{A} \in \mathbb{K}^{m \times N}$ be a matrix satisfying the UP-RIP($2\mathbf{s}, \delta$) where $\delta < 4/\sqrt{41}$.*

Then any uniformly distributed sparse vector $\mathbf{x} \in \Sigma_s^{(\mathcal{W})}$ can be recovered by solving n (BPDN) problems, and the recovery is

$$\widehat{\mathbf{x}} = S^{-1} \sum \widehat{\mathbf{x}^{(i)}}.$$

In addition, let the noise in each (BPDN) problem be controlled by $\|\mathbf{e}^{(i)}\|_2 \leq \eta$, $1 \leq i \leq n$. Then

$$(3.5) \quad \|\widehat{\mathbf{x}} - \mathbf{x}\|_2 \leq \frac{2(2\delta + 1)C^{-1}}{1 - \frac{41}{16}\delta^2} \left(\sum_{i=1}^n \sigma_s(P_i \mathbf{x})_1 + 2n\eta\sqrt{s}\sqrt{1+\delta} \right).$$

PROOF. By (3.4), it follows

$$\begin{aligned}\frac{1+\rho}{1-\rho} &= \frac{1-\delta^2/16+2\delta\sqrt{1-\delta^2}}{1-\frac{41}{16}\delta^2} \\ \frac{\tau}{1-\rho} &= \frac{\sqrt{s}\sqrt{1+\delta}(\sqrt{1-\delta^2}+\frac{5}{4}\delta)}{1-\frac{41}{16}\delta^2}\end{aligned}$$

Due to inequality (3.2), $\|\hat{\mathbf{x}} - \mathbf{x}\|_2$ is bounded from above by

$$(3.6) \quad \frac{2}{C} \sum_{i=1}^n \frac{(1-\delta^2/16+2\delta\sqrt{1-\delta^2})\sigma_s(P_i\mathbf{x})_1 + 2\sqrt{1+\delta}(\sqrt{1-\delta^2}+\frac{5}{4}\delta)\sqrt{s}\eta}{1-\frac{41}{16}\delta^2}$$

Notice that $1-\delta^2/16+2\delta\sqrt{1-\delta^2} \leq 2\delta+1$ when $\delta \in (0, 4/\sqrt{41})$; also, $\sqrt{1-\delta^2}+\frac{5}{4}\delta \leq 1+\frac{5}{4}\delta < 1+2\delta$. Then, (3.6) implies

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_2 \leq \frac{2}{C} \sum_{i=1}^n \frac{(2\delta+1)\sigma_s(P_i\mathbf{x})_1 + 2\sqrt{1+\delta}(2\delta+1)\sqrt{s}\eta}{1-\frac{41}{16}\delta^2}$$

i.e.

$$\|\hat{\mathbf{x}} - \mathbf{x}\|_2 \leq \frac{2(2\delta+1)C^{-1}}{1-\frac{41}{16}\delta^2} \left(\sum_{i=1}^n \sigma_s(P_i\mathbf{x})_1 + 2n\sqrt{s}\eta\sqrt{1+\delta} \right).$$

□

REMARK 3.12. Traditional results in CS theory show a decay of the ℓ^2 error in $s^{-1/2}\sigma_s(\mathbf{x})_1$. This is not the case here. However, such a bound can be achieved, when working with extensions of the ℓ^q robust null space properties (see [19, 4.21] for the single subspace definition). This is actually a passage implicit in the proof of Theorem 3.10 (see for instance [19, Theorem 4.22] with the appropriate changes). We choose not to detail these results to ease the presentation.

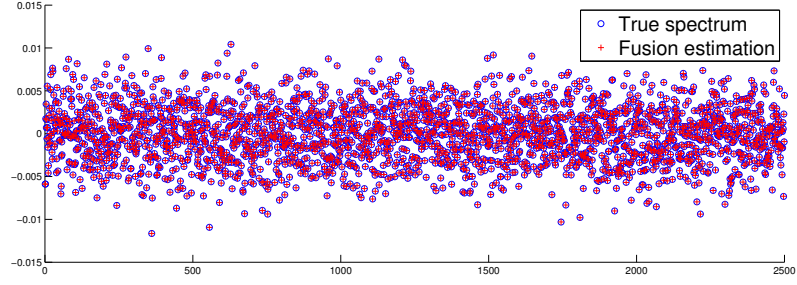
4. Application to dense spectrum estimation

We derive here a simple, yet convincing, example of the application of our approach to the estimation of a dense Fourier spectrum from relatively cheap sensors.

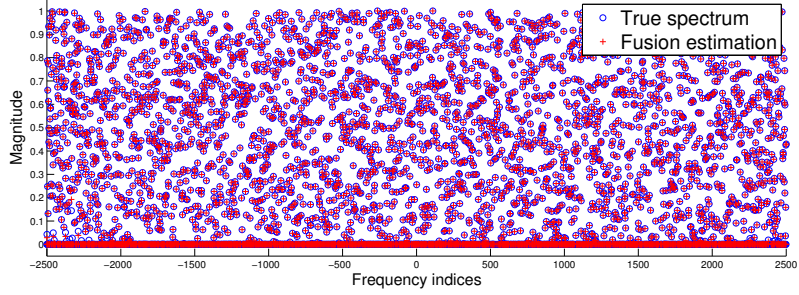
We artificially construct a signal whose Fourier spectrum is sparse. To this end, we consider an $2N+1$ -spectrum with $N = 2500$. 60% of these frequencies (i.e. 3000 indices) are selected at random and the magnitude is set uniformly at random in $[0, 1]$. An example of such a random spectrum is illustrated as the blue dots on Figure 2b.

We then split the spectrum in $n = 60$ equally sized frequency bands, such that each frequency is present in at least 3 bands. The set of projections in the fusion frame approach is then simply a direct band-pass filter. The associated frame operator is hence $S = 3\mathbf{I}$. The measurements are taken via a random subsampled Fourier matrix. Consider the unscaled Fourier matrix $\mathcal{F}_N := \left(e^{\frac{j2\pi kt}{N}} \right)_{-N \leq t, k \leq N}$ and extract m rows indexed by $\Omega \subset \{1, \dots, 2N+1\}$ at random and construct the sensing matrix as $\mathbf{A} = \frac{1}{\sqrt{m}} (\mathcal{F}_N(i, k))_{i \in \Omega, 1 \leq k \leq 2N+1}$. While the optimal bound known so far for RIPs for Fourier matrices [20] suggest that m should scale as $m_{\text{theoretical}} \geq Cs \log(s)^2 \log(N)$, we have here set

$$m = 2s \log(2N+1),$$



(A) Signal generated with a dense Fourier spectrum

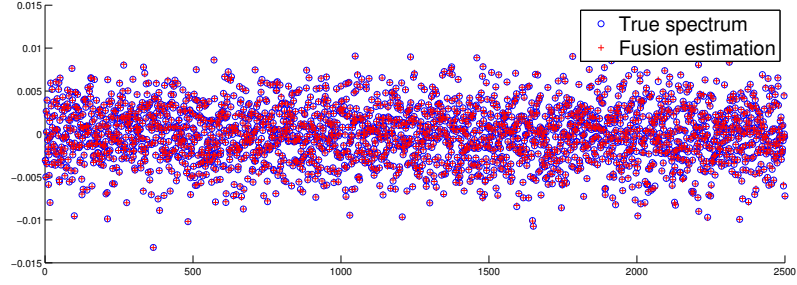


(B) Fourier spectrum: the support and the magnitudes are generated uniformly at random.

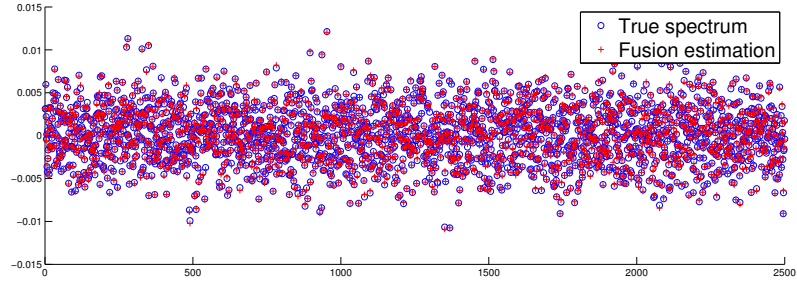
FIGURE 2. Noiseless recovery via the fused compressed sensing approach introduced. Blue circles are true values, while red crosses represent our estimations.

which is believed to be closer to the optimal bounds, and still numerically performs well. It corresponds in this situation to $m = 1197$ measurements per sensor. The sparse recovery is approximated by means of the HTP algorithm [18] to speed up slightly the calculations. The recovered spectrum after n local recovery and a fusion of the data can be seen as the red crosses on Figure 2b. This corresponds to the signal depicted in Figure 2a, where, once again, the blue circles corresponds to the true signal and the red crosses to the one generated after a Fast Fourier Transform on the recovered spectrum.

Lastly Figure 3 shows the result of our fused compressed sensing approach when independent additive Gaussian noise is included in the measurement process. As can be justified by our theorems, the approach is robust, even when fairly strong Gaussian noise is added. Figure 3b includes noise with 0.02 variance indepently at each sensor and yet yields almost perfect recovery of the signal. It is important to notice here that the sensors can be considered *very cheap* in the sense that they account for around 40% of the whole signal size, which in turns means that they also represent only 2/3 of the number of non-zeroes in the spectrum. In this particular case, there is no hope to recover the underlying spectrum without multiple measurements, and the use of the Fusion Frame approach allows for efficient computations and recovery without facing further problems.



(A) Noisy signal recovery with additive Gaussian noise with variance 0.01.



(B) Noisy signal recovery with additive Gaussian noise with variance 0.02.

FIGURE 3. Noisy signal recovery via our fused compressed sensing approach. Top corresponds to .01 additive Gaussian random noise to each sensor, while the bottom one corresponds to .02 additive Gaussian noise.

5. Conclusion

In applications, it is beneficial to increase the number of non-zeros that can be recovered, and the related number of measurements scales with the sparsity/density. To avoid this explosion in sizes, we have combined ideas from distributed compressed sensing and fusion frames reconstruction methods to recover a signal $\mathbf{x} \in \mathbb{K}^N$ from the (noisy) measurements $\mathbf{y}^{(i)} = \mathbf{A}P_i\mathbf{x} + \mathbf{e}^{(i)}$, where the $\{P_i\}_{i=1}^n$ is a family of n projections in \mathbb{K}^N . We have solved this high-dimensional problem $\mathbf{y} = \mathbf{A}\mathbf{x}$ by combining results obtained from n localized problems, by means of fusion frames. Each localized problem is significantly easier to solve. We have shown that our approach extends classical theories in compressed sensing to a distributed sampling setting. In particular, it allows to give a strong mathematical foundations to the engineering problems that may require approaches similar to the ones we described here. One can assume non-uniform sparsity patterns, and nonuniform constants in to expand Definitions 3.1, 3.3 and 3.4; such generalizations are explored in [1]. In particular, by fitting the projections appropriately to the sought to recover signal, one can easily analyze the block sparse model and the t -separation model. One may also be interested in ℓ^1 analysis models, which are investigated in [1]. In particular, this becomes handy when dealing with expansions in coherent frames.

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References

1. Roza Aceska, Jean-Luc Bouchot, and Shidong Li, *Local sparsity and recovery of fusion frames structured signals*, arXiv preprint arXiv:1604.00424, 2017.
2. Ulaş Ayaz, Sjoerd Dirksen, and Holger Rauhut, *Uniform recovery of fusion frame structured sparse signals*, Applied and Computational Harmonic Analysis **41** (2016), no. 2, 341–361.
3. Afonso S. Bandeira, Katya Scheinberg, and Luis Nunes Vicente, *On partial sparse recovery*, arXiv preprint arXiv:1304.2809, 2013.
4. Richard Baraniuk, Mark Davenport, Ronald DeVore, and Michael Wakin, *A simple proof of the restricted isometry property for random matrices*, Constructive Approximation **28** (2008), no. 3, 253–263.
5. Thomas Blumensath and Mike E Davies, *Iterative hard thresholding for compressed sensing*, Applied and Computational Harmonic Analysis **27** (2009), no. 3, 265–274.
6. Jean-Luc Bouchot, *A generalized class of hard thresholding algorithms for sparse signal recovery*, Approximation Theory XIV: San Antonio 2013, Springer, 2014, pp. 45–63.
7. Jean-luc Bouchot, Simon Foucart, and Pawel Hitczenko, *Hard thresholding pursuit algorithms: number of iterations*, Applied and Computational Harmonic Analysis **41** (2016), no. 2, 412–435.
8. Petros Boufounos, Gitta Kutyniok, and Holger Rauhut, *Sparse recovery from combined fusion frame measurements*, Information Theory, IEEE Transactions on **57** (2011), no. 6, 3864–3876.
9. Claire Boyer, Jérémie Bigot, and Pierre Weiss, *Compressed sensing with structured sparsity and structured acquisition*, Applied and Computational Harmonic Analysis (2017).
10. Jameson Cahill, Peter G. Casazza, and Shidong Li, *Non-orthogonal fusion frames and the sparsity of fusion frame operators*, Journal of Fourier Analysis and Applications **18**, no. 2 (2012), 287 – 308.
11. Emmanuel J. Candès, *The restricted isometry property and its implications for compressed sensing*, Comptes Rendus Mathématique **346** (2008), no. 9, 589–592.
12. Peter G. Casazza and Gitta Kutyniok, *Frames of subspaces*, Contemporary Mathematics **345** (2004), 87–113.
13. Peter G. Casazza, Gitta Kutyniok, and Shidong Li, *Fusion frames and distributed processing*, Applied and Computational Harmonic Analysis **25** (2008), 114–132.
14. Ole Christensen, *An introduction to frames and Riesz bases*, Springer Science & Business Media, 2003.
15. Il Yong Chun and Ben Adcock, *Uniform recovery from subgaussian multi-sensor measurements*, 2016.
16. ———, *Compressed sensing and parallel acquisition*, IEEE Transactions on Information Theory **PP** (2017), no. 99, 1–1.
17. Massimo Fornasier and Holger Rauhut, *Compressive sensing*, Handbook of Mathematical Methods in Imaging (2011), 187–228.
18. Simon Foucart, *Hard thresholding pursuit: an algorithm for compressive sensing*, SIAM Journal on Numerical Analysis **49** (2011), no. 6, 2543–2563.
19. Simon Foucart and Holger Rauhut, *A mathematical introduction to compressive sensing*, Springer, 2013.
20. Ishay Haviv and Oded Regev, *The restricted isometry property of subsampled fourier matrices*, Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, 2016, pp. 288–297.
21. Deanna Needell and Joel A. Tropp, *Cosamp: Iterative signal recovery from incomplete and inaccurate samples*, Applied and Computational Harmonic Analysis **26** (2009), no. 3, 301–321.

- 22. Holger Rauhut, *Compressive sensing and structured random matrices*, Theoretical foundations and numerical methods for sparse recovery **9** (2010), 1–92.
- 23. Tong Zhang, *Sparse recovery with orthogonal matching pursuit under rip*, Information Theory, IEEE Transactions on **57** (2011), no. 9, 6215–6221.

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